The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011–2012**

Vector spaces and Fourier theory **2 hours**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Let V be a vector space, and let $v_1, \ldots, v_n \in V$. Define what it means for $v_1, \ldots, v_n$ to

(a) be linearly independent;

(b) span V;

(c) form a basis for V. ***(6 marks)***

(ii) Let $V = \{f \in C^\infty(\mathbb{R}) : f + f'' = 0\}$ and $W = \{f \in C^\infty(\mathbb{R}) : f' + f''' = 0\}$. You may assume that V and W are subspaces of $C^\infty(\mathbb{R})$.

(a) Let $f \in V$, let $a = f'(0)$ and let $b = f(0)$. Let

$$g(x) = f(x) - a\sin(x) - b\cos(x)$$

and let

$$h(x) = g(x)^2 + g'(x)^2.$$

Show that $g \in V$ and that $g(0) = g'(0)$. Show also that $h(0) = 0$ and $h' = 0$. Deduce that $h = 0$ and that $g = 0$. Hence show that sin and cos span V. ***(11 marks)***

(b) Show that $V \subseteq W$ and that if $f \in W$ then $f' \in V$. ***(2 marks)***

(c) Let $j : \mathbb{R} \to \mathbb{R}$ be the constant function such that $j(x) = 1$ for all $x \in \mathbb{R}$. Show that $j \in W$ and that sin, cos and j form a basis for W. ***(6 marks)***

2 Let V be a vector space and let U and W be subspaces of V.

(i) Show that $U \cap W$ is a subspace of V. ***(5 marks)***

Define the sum $U + W$ and show that $U \subseteq U + W$ and $W \subseteq U + W$. ***(3 marks)***

(ii) Write down, without proof,

(a) an inequality relating $\dim U$ and $\dim V$; ***(1 mark)***

(b) a strong conclusion about U and V whenever $\dim(U) = \dim(V)$; ***(1 mark)***

(c) a formula relating $\dim(U + W)$, $\dim(U \cap W)$, $\dim(U)$ and $\dim(W)$. ***(1 mark)***

(iii) Suppose that $\dim(V) = 3$, that $\dim U = \dim W = 2$ and that $U \neq W$. Show that

(a) $U + W = V$; ***(4 marks)***

(b) $\dim(U \cap W) = 1$. ***(2 marks)***

(iv) Let

$$\begin{aligned} V &= \{(a, b, c, d)^T \in \mathbb{R}^4 | a + b + c + d = 0\} \\ U &= \{(a, b, c, d)^T \in \mathbb{R}^4 | a + b + c + d = 0 = a + b\} \text{ and} \\ W &= \{(a, b, c, d)^T \in \mathbb{R}^4 | a + b + c + d = 0 = a - c\}. \end{aligned}$$

You may assume that these are all subspaces of $\mathbb{R}^4$.

(a) Write down, without proof, $\dim(V)$, $\dim(U)$ and $\dim(W)$. Find a vector $v \in \mathbb{R}^4$ that spans $U \cap W$. ***(4 marks)***

(b) Let $u = (2, -2, 7, -7)$ and let $w = (3, -5, 3, -1)$. Find $u' \in U$ and $w' \in W$ such that $u' + w' = u + w$ but $u' \neq u$. ***(4 marks)***

3 (i) Let V and W be vector spaces and let $\phi : V \to W$ be a linear map. Define the *kernel* $\ker(\phi)$ and the *image* $\mathrm{im}(\phi)$. ***(2 marks)***

State, without proof, a formula relating $\dim(\ker(\phi))$, $\dim(\mathrm{im}(\phi))$ and $\dim(V)$. ***(1 mark)***

(ii) Let $V = M_2(\mathbb{R})$ be the vector space of 2×2 real matrices, let

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$$

and let $\phi : V \to V$ be the map such that $\phi(X) = AX - XA$ for all $X \in V$. You may assume that ϕ is linear.

(a) Find the matrix of ϕ with respect to the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(8 marks)

(b) Show that $I_2 \in \ker(\phi)$ and that $A \in \ker(\phi)$. Deduce that $\dim(\ker(\phi)) \geq 2$ and $\dim(\mathrm{im}(\phi)) \leq 2$. ***(5 marks)***

Write down two linearly independent elements of $\mathrm{im}(\phi)$ and show that $\dim(\mathrm{im}(\phi)) = 2 = \dim(\ker(\phi))$. ***(3 marks)***

(c) Let W be the vector space of 2×2 real matrices with trace 0. Show that $\mathrm{im}(\phi) \subseteq W$. Is $\mathrm{im}(\phi) = W$? Justify your answer. ***(3 marks)***

(d) Show that $A^2 \in \ker(\phi)$ and express A^2 as a linear combination of I_2 and A. ***(3 marks)***

4 (i) Define the notion of an *inner product* on a finite-dimensional vector space over $\mathbb{R}$. ***(5 marks)***

(ii) Let V be an inner product space (over $\mathbb{R}$). State the Cauchy-Schwarz inequality for vectors $v, w \in V$, including the criterion for when equality holds. ***(3 marks)***

(iii) Consider the inner product space $C[0,1]$ with inner product given by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx.$$

(a) Show that for any $f \in C[0,1]$, we have

$$\left| \int_0^1 (1 - x^3) f(x)\,dx \right| \leq \frac{3}{\sqrt{14}} \sqrt{\int_0^1 f(x)^2\,dx}.$$ ***(6 marks)***

(b) Use the Gram-Schmidt process to find an orthogonal basis for the subspace of the inner product space $C[0,1]$ spanned by 1, x and x^4. ***(11 marks)***

5 (i) Let V be a vector space with an inner product and let $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ be a finite subset of V. Explain what it means for $\mathcal{V}$ to be *orthogonal* and what it means for $\mathcal{V}$ to be *strictly orthogonal*. ***(3 marks)***

Show that if $\mathcal{V}$ is strictly orthogonal then $v_1, v_2, \ldots, v_n$ are linearly independent. ***(5 marks)***

(ii) Consider the Fourier inner product space $C[-\pi, \pi]$ of continuous functions $[-\pi, \pi] \to \mathbb{R}$ with the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)g(t)\, dt.$$

Let $f(t) = \sin t \cos 5t$ and $g(t) = \sin 2t \cos 4t$. Calculate the angle between f and g and the distance between f and g. ***(11 marks)***

(iii) Consider the subspace V of the Fourier inner product space of trigonometric polynomials of degree at most 1, spanned by the set $\mathcal{V} = \{1, \cos t, \sin t\}$, and the space $M_2(\mathbb{R})$ of 2×2 real matrices with inner product $\langle A, B \rangle = \text{trace}(A^T B)$. If $v = \alpha + \beta \cos t + \gamma \sin t$, define $\phi(v) = \begin{pmatrix} \beta & \alpha \\ 0 & \gamma \end{pmatrix}$.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, compute $\langle \phi(v), A \rangle$, and hence give the adjoint map $\hat{\phi} : M_2(\mathbb{R}) \longrightarrow V$. ***(6 marks)***

End of Question Paper