The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011–2012**

Nonlinear Mathematics **2 hours**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Let $\varphi\colon P \to Q$ be a map where P and Q are subsets of $\mathbb{R}^3$.

(a) For $(u, v, w) \in Q$ define the preimage $\varphi^{-1}(u, v, w)$. **(2 marks)**

Now define $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $\varphi(x, y, z) = (yz,\ zx,\ xy)$. Find each of the following preimages:

(b) $\varphi^{-1}(0, 0, 0)$

(c) $\varphi^{-1}(1, 0, 0)$

(d) $\varphi^{-1}(1, 1, 0)$

(e) $\varphi^{-1}(1, 1, 1)$ **(13 marks)**

(ii) (a) Let $(a, b) \in \mathbb{R}^2$ and let $r > 0$. Define the *open ball* $B((a, b), r)$.

Define what it means for a set $P \subseteq \mathbb{R}^2$ to be *open*. **(5 marks)**

(b) Let $a \in \mathbb{R}$ and let $P = \{(x, y) \in \mathbb{R}^2 \mid x > a\}$. Prove *directly from your definition in* (ii)(a) that P is an open set. **(5 marks)**

2 (i) Consider a function $F\colon\ P \to \mathbb{R}$ defined on an open set $P \subseteq \mathbb{R}^2$. Define what it means for F to be *continuous* at $(a,b) \in P$.

Now consider a map $\varphi\colon\ P \to \mathbb{R}^2$ defined on an open set $P \subseteq \mathbb{R}^2$. Define what it means for φ to be *continuous* at $(a,b) \in P$. **(5 marks)**

(ii) (a) Let $\varphi\colon\ P \to \mathbb{R}^2$ be a map defined on an open set $P \subseteq \mathbb{R}^2$, and assume that φ is continuous at $(a,b) \in P$. Show that for each $\varepsilon > 0$ there is an $r > 0$ such that

$$\text{if} \quad (x,y) \in B((a,b),r) \quad \text{then} \quad \varphi(x,y) \in B(\varphi(a,b),\varepsilon).$$

(10 marks)

(b) Let P and Q be open sets in $\mathbb{R}^2$, let $\varphi\colon\ P \to \mathbb{R}^2$ be a map, and let $H\colon\ Q \to \mathbb{R}$ be a function defined on an open set $Q \subseteq \mathbb{R}^2$ such that $\varphi(x,y) \in Q$ for all $(x,y) \in P$, so that $H \circ \varphi$ is defined.

Using (a), or otherwise, show that if φ is continuous at $(a,b) \in P$ and H is continuous at $\varphi(a,b)$, then the composite $H \circ \varphi$ is continuous at (a,b). **(10 marks)**

3 (i) Consider a function $F\colon\ P \to \mathbb{R}$ defined on an open set $P \subseteq \mathbb{R}^2$. Define what it means for F to be C^1 on P. **(3 marks)**

(ii) (a) Let $\varphi\colon\ P \to \mathbb{R}^2$ be a map defined on an open set $P \subseteq \mathbb{R}^2$, and let $\psi\colon\ Q \to \mathbb{R}^2$ be a map defined on an open set $Q \subseteq \mathbb{R}^2$. Suppose that $\varphi(x,y) \in Q$ for all $(x,y) \in P$ so that the composite $\psi \circ \varphi$ is defined.

State the Chain Rule, in concise form, for $\psi \circ \varphi$. **(4 marks)**

(b) Define what it means for $\varphi\colon\ P \to Q$ to be a diffeomorphism. **(3 marks)**

(c) Show that if $\varphi\colon\ P \to Q$ is a diffeomorphism then $\det D(\varphi)(x,y) \neq 0$ for all $(x,y) \in P$. **(5 marks)**

(d) For φ a diffeomorphism, write down the matrix for $D(\varphi^{-1})(u,v)$ in terms of partial derivatives of φ. **(4 marks)**

(e) Define sets $P = \{(x,y) \in \mathbb{R}^2 \mid x > 0,\ y > 0,\ x + y < \frac{\pi}{2}\}$, and $Q = \{(u,v) \in \mathbb{R}^2 \mid u > 0,\ v > 0\}$.

The map $\varphi\colon\ P \to Q$ defined by

$$\varphi(x,y) = \left(\frac{\sin x}{\cos y},\ \frac{\sin y}{\cos x}\right).$$

is known to be a diffeomorphism. Write down, in terms of x and y, the matrix for $D(\varphi^{-1})(\varphi(x,y))$ for $(x,y) \in P$. **(6 marks)**

4 (i) Define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by $F(x,y) = \sqrt{x^4 + y^4}$. Prove in detail that F is C^1 on $\mathbb{R}^2$. **(11 marks)**

(ii) Consider a C^1 map $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$, and write

$$\varphi(x,y) = (F(x,y),\, G(x,y))$$

as usual. Assume there is a C^1 map $H\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$u = F(x,y) \quad \text{if and only if} \quad x = H(u,y).$$

(a) Show that

$$\frac{\partial F}{\partial x}\frac{\partial H}{\partial u} = 1 \quad \text{and} \quad \frac{\partial F}{\partial x}\frac{\partial H}{\partial y} + \frac{\partial F}{\partial y} = 0.$$

(4 marks)

(b) Now define $K\colon \mathbb{R}^2 \to \mathbb{R}$ by $K(u,y) = G(H(u,y),y)$ and assume that, for all $(u,y) \in \mathbb{R}^2$,

$$\frac{\partial H}{\partial u}(u,y) = \frac{\partial K}{\partial y}(u,y) \neq 0.$$

Show that $\det D(\varphi)(x,y) = 1$ for all $(x,y) \in \mathbb{R}^2$. **(10 marks)**

5 (i) State carefully and in full the Local Diffeomorphism Theorem for maps $\varphi\colon P \to Q$ where P and Q are open sets in $\mathbb{R}^3$. **(6 marks)**

(ii) (a) Let $Q \subseteq \mathbb{R}^3$ be an open set and let $(a,b,c) \in Q$. Show that there is an open interval $J \subseteq \mathbb{R}$ containing c such that $(a,b,z) \in Q$ for all $z \in J$. **(3 marks)**

Now let $\varphi\colon P \to \mathbb{R}^2$ be a C^1 map defined on an open set $P \subseteq \mathbb{R}^3$ and write

$$\varphi(x,y,z) = (F(x,y,z),\, G(x,y,z))$$

as usual. Define $\Phi\colon P \to \mathbb{R}^3$ by

$$\Phi(x,y,z) = (F(x,y,z),\, G(x,y,z),\, z).$$

(b) Find $D(\Phi)(x,y,z)$ in terms of partial derivatives of F and G. **(4 marks)**

(c) Let $(x_0,y_0,z_0) \in P$ be such that $F(x_0,y_0,z_0) = 0$ and $G(x_0,y_0,z_0) = 0$.

Suppose that $P_1 \subseteq P$ is an open set containing (x_0,y_0,z_0) and $Q_1 \subseteq \mathbb{R}^3$ is an open set containing $\Phi(x_0,y_0,z_0)$, such that $\Phi(P_1) = Q_1$ and the restriction $\Phi\colon P_1 \to Q_1$ is a diffeomorphism.

Write $\psi = \Phi^{-1}\colon Q_1 \to P_1$ for the inverse of $\Phi\colon P_1 \to Q_1$ and write ψ_1, ψ_2, ψ_3 for the components of ψ. Prove that, for all $(u,v,w) \in Q_1$,

$$F(\psi_1(u,v,w),\, \psi_2(u,v,w),\, w) = u,$$
$$G(\psi_1(u,v,w),\, \psi_2(u,v,w),\, w) = v.$$

(6 marks)

(d) You may assume that there is an open set $W \subseteq \mathbb{R}^2$ containing (x_0,y_0) and an open interval $I \subseteq \mathbb{R}$ containing z_0 such that $W \times I \subseteq P_1$. For such a W and I, define functions $\alpha_1\colon I \to \mathbb{R}$ and $\alpha_2\colon I \to \mathbb{R}$ such that, for all $z \in I$,

$$F(\alpha_1(z), \alpha_2(z), z) = 0, \quad \text{and} \quad G(\alpha_1(z), \alpha_2(z), z) = 0,$$

and such that if $(x,y,z) \in W \times I$ and $\varphi(x,y,z) = (0,0)$, then $x = \alpha_1(z)$ and $y = \alpha_2(z)$. **(6 marks)**

End of Question Paper