Data provided: Formulae sheet

The University Of Sheffield.

CIV340

SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester 2011-2012

Computational Engineering Mathematics

Three hours

Marks will be awarded for your best FOUR answers

1 (i) A second order PDE for the function $U(x,y)$ can be written as

$$A\frac{\partial^2 U}{\partial x^2} + B\frac{\partial^2 U}{\partial x \partial y} + C\frac{\partial^2 U}{\partial y^2} + D\frac{\partial U}{\partial x} + E\frac{\partial U}{\partial y} + FU = 0,$$

where A, B, C, D, E and F are constants or functions of x, y. This equation can be classified as being either elliptic, parabolic or hyperbolic according to the values of A, B and C. State the criterion for the classification and hence classify each of the following PDEs:

(a) $\dfrac{\partial^2 U}{\partial x^2} + 4\dfrac{\partial^2 U}{\partial x \partial y} - \dfrac{\partial U}{\partial y} = U \sin x$;

(b) $\dfrac{\partial^2 U}{\partial x^2} + 6\dfrac{\partial^2 U}{\partial x \partial y} + 9\dfrac{\partial^2 U}{\partial y^2} = \dfrac{\partial U}{\partial x} + 5x^2$;

(c) $-\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial x \partial y} = \dfrac{\partial^2 U}{\partial y^2}$.

(d) $(x+1)\dfrac{\partial^2 U}{\partial x^2} - (x+1)\dfrac{\partial^2 U}{\partial x \partial y} + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial y^2} = 0.$

(7 marks)

(ii) The one-dimensional diffusion equation is given by

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2} \quad (0 \le x \le 1), \qquad (1)$$

where α is the diffusion coefficient. The equation is to be solved together with the necessary additional conditions:

$$U(x,0) = f(x) \quad \text{and} \quad U(0,t) = a, \; U(1,t) = b,$$

where we use the standard notation $U_{ij} \equiv U(x_i, t_j)$ and the conventions that $i = 0$ and $i = N$ correspond to $x = 0$ and $x = 1$, respectively, and $j = 0$ corresponds to $t = 0$. Use suitable finite difference approximations, given on the formulae sheet, to derive the *implicit scheme*

$$\alpha k U_{i+1,j} - (1 + 2\alpha k)U_{i,j} + \alpha k U_{i-1,j} = -U_{i,j-1},$$

for $i = 1, .., N-1$, $j = 1, 2, ..$, where $k = \Delta t/(\Delta x)^2$. ***(5 marks)***

(iii) The diffusion equation (equation 1) is to be solved (approximately) over the range $0 \le x \le 1$ for the temperature distribution along a given steel billet with boundary conditions $U(0,t) = 20^oC$ and $U(1,t) = 10^oC$ and initial conditions $U(x,0) = 20 - 10x$, where it is assumed that the units have been normalized so that $\alpha = 1$. Assuming that we use $\Delta x = 0.25$ and $\Delta t = 0.2$, then use the *implicit* scheme to write down the system of algebraic equations for the temperature at $x = 0.25, 0.5, 0.75$ and time $t = 0.2$. Do NOT attempt to solve the system. ***(10 marks)***

 Question 1 continued on next page

1 (continued)

(iv) Use the same notations as in part (ii) to show that the *implicit scheme* for the equation

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2} + \beta \frac{\partial U}{\partial x}$$

is

$$(\alpha k + \beta h) U_{i+1,j} - (1 + 2\alpha k) U_{i,j} + (\alpha k - \beta h) U_{i-1,j} = -U_{i,j-1},$$

where β is a constant, $h = \Delta t/(2\Delta x)$ and $k = \Delta t/(\Delta x)^2$. ***(3 marks)***

2 (i) The elastic constitutive matrix for an isotropic material is given by

$$[C] = \begin{bmatrix} (\lambda+2\mu) & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & (\lambda+2\mu) & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & (\lambda+2\mu) & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix}$$

where

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

Given further that $E = 3.32$ GPa, $\nu = 0.178$, and that a state of strain defined by $\varepsilon_{xx} = 0.7013$, $\varepsilon_{yy} = -0.31\varepsilon_{xx}$, $\varepsilon_{zz} = -0.19\varepsilon_{xx}$, $\varepsilon_{xy} = 0.2274$, $\varepsilon_{yz} = 0.6413$ and $\varepsilon_{zx} = -0.8273$ exists at a point in a given isotropic material, calculate the corresponding state of stress at the point. Work correct to four decimal places. ***(11 marks)***

(ii) Using the results in the first part, evaluate the following quantities:

- The stress force on a unit area element in the direction $\mathbf{n} = (0.7071, -0.7071, 0)^T$;
- The mean stress $\sigma_{kk}/3$;
- $\sigma_{3k}\varepsilon_{k2}$.

Note that $\varepsilon_{11} = \varepsilon_{xx}$, $\varepsilon_{12} = \varepsilon_{xy}$, etc. ***(9 marks)***

(iii) The deviatoric stress is defined by

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk},$$

where δ_{ij} is the Kronecker delta tensor. Show that $S_{ii} = 0$, and that

$$S_{ij}\varepsilon_{ij} = \sigma_{ij}\varepsilon_{ij} - \frac{1}{3}\varepsilon_{mm}\sigma_{kk}.$$

What is the order of $S_{ij}\varepsilon_{ij}$ as a tensor expression? ***(5 marks)***

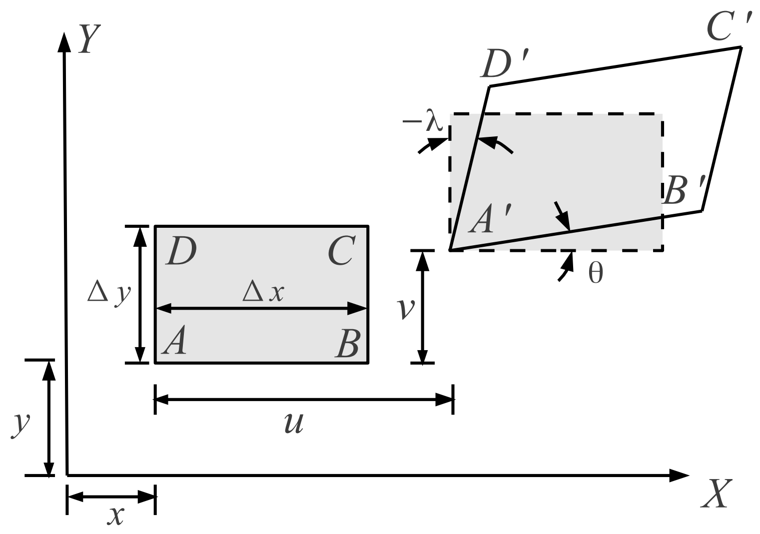

Figure 1: Illustraion of two dimensional strain.

3 (i) Referring to Figure 1, define the normal strain in y direction, ε_{yy}, and the engineering shear strain, γ_{xy}, hence show that

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \lambda = -\frac{\partial u}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

(19 marks)

(ii) The velocity field in an unsteady moving fluid is given by $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, where $u \equiv u(x, y, z, t)$, $v \equiv v(x, y, z, t)$ and $w \equiv w(x, y, z, t)$. The divergence of $\mathbf{V}$ is defined by

$$\nabla \cdot \mathbf{V} \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \lim_{\Delta\mathcal{V} \to 0} \frac{1}{\Delta\mathcal{V}} \frac{D(\Delta\mathcal{V})}{Dt},$$

where $\Delta\mathcal{V}$ is the volume of a moving fluid element, and D/Dt is the substantial derivative. By considering the mass of the fluid element, $\Delta m = \rho\Delta\mathcal{V}$, derive the equation of continuity for a compressible fluid and show that this can be rearranged as:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0\ .$$

(6 marks)

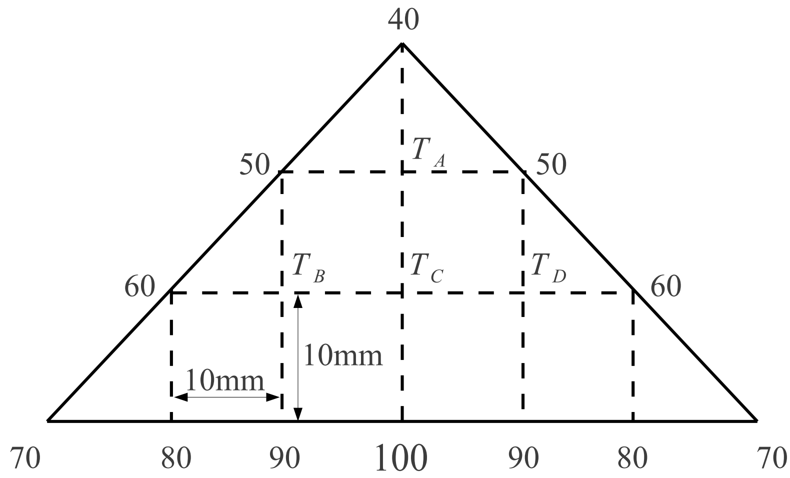

Figure 2: A triangular plate with temperature defined on the boundaries.

4 Figure 2 shows a triangular plate made of an homogeneous isotropic material. The temperature distribution in this plate satisfies the indicated boundary conditions (given in degrees centigrade) and has reached a steady-state condition so that the temperature is described by Laplace's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

(i) Draw a sketch of the solution domain showing clearly the line of symmetry for the temperature distribution and indicate which of the unknown temperatures are equal to each other. ***(3 marks)***

(ii) Use the finite difference formulae on the formulae sheet to formulate the finite difference equations required to find estimates of the nodal temperatures, T_A, T_B and T_C. ***(9 marks)***

(iii) Derive the Jacobi iteration formulae to find the $(k+1)$th iteration for T_A, T_B and T_C from the kth iteration. ***(13 marks)***

5 (i) The boundary value problem

$$\mathcal{L}(u) \equiv \frac{d^2u}{dx^2} - 9u - 3x^2 = 0, \text{ given } u(0) = 0 \text{ and } u(1) = 1$$

is to be solved by the weighted residual method.

(a) Determine which of the following $U_1(x)$ and $U_2(x)$ can be taken as the trial function:

$$U_1(x) = (1 - x) + c_1 x(1 - x) + c_2 x(1 - x^2)$$

$$U_2(x) = x + c_1(x^2 - x) + c_2(x^3 - x)$$

(b) Determine the residual $\mathcal{L}(U) = R(x)$ associated with your chosen trial function. Use the condition

$$\int_0^1 w(x)R(x)dx = 0$$

to derive two equations for c_1 and c_2, with the weight functions $w(x) = 1$ and $w(x) = x$, respectively. Solve these equations working correct to four decimal places.

(16 marks)

(ii) Given the ordinary differential equation

$$\frac{d}{dx}\left(A(x)\frac{du}{dx}\right) + f(x) = 0, \quad 0 \le x \le 1,$$

where $A(x)$ and $f(x)$ are known functions, and $u(x)$ to be determined, show that the weak form of the equation is

$$\int_0^1 A\frac{dw}{dx}\frac{du}{dx}dx = w(1)F(1) - w(0)F(0) + \int_0^1 w(x)f(x)dx,$$

where $F(x) \equiv A(x)du(x)/dx$ and $w(x)$ is the weight function. ***(9 marks)***

End of Question Paper

Formulae Sheet

Notation:

$$U(x_i,\ t_j) \equiv U_{i\,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1} - U_{ij}}{\Delta t}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{ij} - U_{i,j-1}}{\Delta t}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{\Delta x^2}$$