The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–12**

INTRODUCTION TO RELATIVITY

2 hours

Marks will be awarded for your best ***FOUR*** *answers.*

1 (i) Define what is meant by an *inertial frame*.

Two inertial frames, $R : (ct, x)$ and $\tilde{R} : (c\tilde{t}, \tilde{x})$ are related by the *Lorentz transformation*

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \end{pmatrix} = \gamma(u) \begin{pmatrix} 1 & -\dfrac{u}{c} \\ -\dfrac{u}{c} & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix},$$

where

$$\gamma(u) = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}.$$

Verify that

$$c^2\tilde{t}^2 - \tilde{x}^2 = c^2t^2 - x^2.$$

Show that a light ray passing through the origin of the inertial frame R and travelling with speed c along the x-axis of the inertial frame R (in the direction of increasing x) also travels with speed c in the inertial frame $\tilde{R}$.

(10 marks)

(ii) Now consider a second light ray passing through the point $x = D$ at time $t = 0$ in the frame R, and travelling with speed c along the x-axis in the direction of increasing x.

Write down the path of the second light ray in the frame R and show that the path of this second light ray in the frame $\tilde{R}$ is

$$\gamma(u) \begin{pmatrix} ct - ut - \dfrac{uD}{c} \\ -ut + ct + D \end{pmatrix}.$$

At time $\tilde{T}$ in the frame $\tilde{R}$, show that the second light ray is at the point

$$\tilde{x} = c\tilde{T} + \gamma(u)D\left(1 + \frac{u}{c}\right).$$

Hence show that, at time $\tilde{T}$ in the frame $\tilde{R}$, the light rays are a distance L apart, where

$$L = D\sqrt{\frac{c+u}{c-u}}.$$

(15 marks)

2 (i) Write down the (4×4) matrix which is the *metric tensor* g, and show that $g^2 = I$, the (4×4) identity matrix.

Two inertial frames $R : (ct, x, y, z)$ and $\tilde{R} : (c\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ are related by the transformation

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = L \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \qquad (*)$$

where L is a (4×4)-matrix with constant entries.

Write down the conditions on the matrix L for the transformation $(*)$ to be a *proper orthochronous Lorentz transformation.* Suppose that the matrices L and M represent proper, orthochronous Lorentz transformations. Show that the matrix LM also represents a proper Lorentz transformation.

(11 marks)

(ii) Suppose that the matrices L and M represent proper, orthochronous Lorentz transformations and that the matrix M has the form

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & M_1^1 & M_2^1 & M_3^1 \\ 0 & M_1^2 & M_2^2 & M_3^2 \\ 0 & M_1^3 & M_2^3 & M_3^3 \end{pmatrix}.$$

Show that the matrices LM and ML represent orthochronous Lorentz transformations. Suppose that the matrix L is symmetric, so that $L^T = L$, and that L represents a proper, orthochronous Lorentz transformation.

Show that the matrix L has inverse

$$N = gLg$$

and that the matrix N also represents a proper, orthochronous Lorentz transformation.

(14 marks)

3 (i) Let X and Y be any two four-vectors and let $Z = X + Y$.

Show that
$$g(Z, Z) = g(X, X) + 2g(X, Y) + g(Y, Y).$$

Deduce that, if X and Y are both null, then
$$g(Z, Z) = 2g(X, Y).$$

(4 marks)

(ii) For the rest of this question, assume that X and Y are future-pointing null four-vectors.

Show that Z is future-pointing.

Now suppose that there is an inertial frame R in which X has components
$$X = (3, 3, 0, 0),$$

and Y has components
$$Y = (\alpha, \beta, \gamma, \delta)\,.$$

Show that, in the inertial frame R,
$$g(X, Y) = 3(\alpha - \beta).$$

Use the fact that Y is a null four-vector to show that
$$\alpha^2 - \beta^2 \geq 0.$$

(10 marks)

(iii) Use this result to show that $Z = X + Y$ is either timelike or null. Find examples of future-pointing null four-vectors X and Y such that $X + Y$ is (a) timelike, and (b) null. ***(11 marks)***

4 (i) In an inertial frame R, an observer has worldline $X(t) = (ct, x, 0, 0)$. Define the quantities *proper time*, *four-velocity* and *four-acceleration* for this observer. Show that the four-velocity of the observer takes the form

$$V = \gamma(v)\,(c, v, 0, 0)\,,$$

where v is the speed of the observer in the inertial frame $R : (ct, x, y, z)$.

(7 marks)

(ii) The inertial frame R is related to a second inertial frame $\tilde{R} : (c\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ by the Lorentz transformation

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \end{pmatrix} = \gamma(u) \begin{pmatrix} 1 & -\dfrac{u}{c} \\ -\dfrac{u}{c} & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix},$$

with $\tilde{y} = y$ and $\tilde{z} = z$, where

$$\gamma(u) = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}.$$

Show that, along the path of the observer,

$$\frac{d\tilde{t}}{dt} = \gamma(u)\left(1 - \frac{uv}{c^2}\right).$$

Find the components of the four-velocity in the inertial frame $\tilde{R}$, and show that, if $\tilde{v}$ is the speed of the observer in the inertial frame $\tilde{R}$, then

$$\tilde{v} = \frac{v - u}{1 - \frac{uv}{c^2}}.$$

Deduce that there is a value of u for which the observer is at rest in the frame $\tilde{R}$.

(11 marks)

(iii) Let $a = \dfrac{dv}{dt}$ and $\tilde{a} = \dfrac{d\tilde{v}}{d\tilde{t}}$ be the accelerations of the observer in the inertial frames R and $\tilde{R}$ respectively. Show that

$$\tilde{a} = a\left(1 - \frac{uv}{c^2}\right)^{-3} \gamma(u)^{-3}.$$

Deduce that, if $a \neq 0$, there is no value of u for which the observer is not accelerating in the inertial frame $\tilde{R}$.

(7 marks)

5 (i) Define the *rest mass* and *four-momentum* of a particle. ***(3 marks)***

(ii) A particle of rest mass m is at rest in an inertial frame R when it splits into two identical particles, each of which moves off with speed v.

Show that the rest mass of the identical particles is

$$\frac{1}{2}m\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}},$$

and that the identical particles move off in opposite directions.

(10 marks)

(iii) A particle with four-momentum $P = (Mc, \boldsymbol{p})$ collides and coalesces with a stationary particle of rest mass m.

What is the four-momentum of the combined particle after the collision? Show that the velocity of the combined particle after the collision is:

$$\frac{\boldsymbol{p}}{M+m}.$$

Show that the rest mass of the combined particle after the collision is:

$$\sqrt{(M+m)^2-\frac{|\boldsymbol{p}|^2}{c^2}}.$$

(12 marks)

End of Question Paper