The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2011–12

WAVES — 2 hours

Marks will be awarded for your best FOUR answers. The marks awarded to each question or section of question are shown in italics.

1 The one-dimensional wave equation for $\phi(x, t)$ is

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}.$$

(i) Show that the general solution for $\phi\,(x, t)$ is

$$\phi(x, t) = f(x - ct) + g(x + ct),$$

where f and g are arbitrary functions.

(11 marks)

(ii) Given that

$$\phi(x, 0) = \begin{cases} 0 & (-\infty < x \le -a) \\ a + x & (-a \le x \le 0) \\ a - x & (0 \le x \le a) \\ 0 & (a \le x < \infty), \end{cases}$$

and that $\dfrac{\partial \phi(x, 0)}{\partial t} = 0$ for all x, find $\phi(x, t)$ where $a > 0$.

(9 marks)

(iii) Sketch the graph of $\phi\,(x, t)$ against x when $ct = 2a$.

(5 marks)

2 A uniform finite string of length l and mass per unit length ρ occupies the interval $0 \leq x \leq l$ and undergoes transverse vibrations with displacement $y(x,t)$, where $c^2 y_{xx} = y_{tt}$, and c^2 is a constant. The tension in the string is ρc^2. You are given that

(a) $y(0,t) = y(l,t) = 0$;
(b) $y(x,0) = (h/l^2)x(l-x)$ where h is a constant;
(c) $\dot{y}(x,0) = 0$;
(d) $y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi ct}{l}\right)$, where the a_n $(n = 1, 2, 3, ..)$ are constants.

(i) Verify that the series in (d) satisfies the PDE and conditions (a) and (c). Find the a_n so that (b) is satisfied.

(17 marks)

(ii) Deduce that the potential energy stored in the string at time t is

$$\frac{16\rho h^2 c^2}{\pi^4 l} \sum_{m=0}^{\infty} \frac{\cos^2\left\{\frac{(2m+1)\pi ct}{l}\right\}}{(2m+1)^4}.$$

(8 marks)

3 (*A model of a stethoscope.*) Sound waves propagate in the positive Oz direction inside the circular cylinder $r = a$ (where $r^2 = x^2 + y^2$ in standard notation). The velocity potential ϕ satisfies

$$c^2 \left\{ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} \right\} = \frac{\partial^2 \phi}{\partial t^2}, \qquad (1)$$

where the constant c is the speed of sound.

(i) State how c is related to the pressure (p) – density (ρ) relationship. Determine the value of c for the case when this is

$$\left(\frac{p}{p_0}\right) = \left(\frac{\rho}{\rho_0}\right)^{\gamma},$$

where $\gamma = 1.4$, and p_0 and ρ_0 are the ambient pressure and density with $p_0 \approx 1.013 \times 10^5$ N m^{-2}, $\rho_0 \approx 1.293$ kg m^{-3}.

(6 marks)

(ii) Seek solutions of (1) of the form

$$\phi = g(r) \exp\{i(kz - \omega t)\},$$

where k and ω are real positive constants. Show that

$$g''(r) + \frac{1}{r} g'(r) + m^2 g(r) = 0 \qquad (2)$$

where m^2 is a constant, depending on ω, c and k. (You may assume that $m^2 > 0$.)

(7 marks)

 Question 3 continued on next page

3 (continued)

(iii) Given that ϕ is bounded at $r = 0$, that $\dfrac{\partial \phi}{\partial r} = 0$ at $r = a$, and that the only solution of (2) that is bounded at $r = 0$ must be a multiple of $J_0(mr)$, where $J_0(\xi)$ is the Bessel function of order zero, show that $m = m_n$ $(n = 1, 2, \ldots)$, where $m_n = \beta_n/a$ and β_n is the nth non-zero root of $J_0'(\xi) = 0$. Given that the β_n are discrete, that $\beta_1 < \beta_2 < \ldots$, and that $\beta_n \to \infty$ as $n \to \infty$, deduce that, for fixed ω, there are a finite number of positive values of k.

(12 marks)

4 The equilibrium position of the free surface of a liquid of infinite depth is $z = 0$, where z is measured vertically upwards. A surface wave causes the displacement of this surface to be $\eta(x, t)$, where x is measured along the undisturbed surface and

$$\eta = a \sin(kx - \omega t),$$

with a, k and ω being positive constants with a small. You are given that the velocity potential $\phi = \phi(x, z, t)$ satisfies

$$\phi_{xx} + \phi_{zz} = 0.$$

You are also given that (a) $\phi_z \to 0$ as $z \to -\infty$; (b) $\phi_z = \eta_t$ at $z = 0$; (c) $\phi_t + g\eta = 0$ at $z = 0$.

(i) Give a brief physical interpretation of (a), (b) and (c).

(6 marks)

(ii) Find $\phi(x, z, t)$ and show that $\omega^2 = gk$.

(13 marks)

(iii) Determine the phase velocity c and the group velocity c_g in terms of k. Show that $c_g = c/2$, and state two quantities that are propagated with speed c_g.

(6 marks)

5 (i) Solve the equation

$$yz_x + xz_y = xy,$$

given that $z = e^{-y^2}$ on $x = 0$ for $y \geq 0$ and that $z = e^{-x^2}$ on $y = 0$ for $x \geq 0$. $\Big[$*We use the notation* $z_x = \dfrac{\partial z}{\partial x}$ *etc.*$\Big]$

(13 marks)

(ii) On what region D in the x–y plane is the solution unique? Verify that z is everywhere continuous in D, but that z_x and z_y are discontinuous across one curve. Determine the curve of discontinuity.

(7 marks)

(iii) More generally, suppose that within a region D^* in the x–y plane, the solution $z = z(x, y)$ of

$$Pz_x + Qz_y = R,$$

where P, Q, R are continuous functions of x, y, z, is everywhere continuous, but that there may be discontinuities in z_x and z_y across a curve Γ. Show that $dy/dx = Q/P$ on Γ.

(5 marks)

End of Question Paper