The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2011–2012

OPERATIONS RESEARCH — 2 Hours

*Marks will be awarded for your best **four** answers.*

1 (i) A company plans to build **seven** factories on selected sites during the next **five** years. Let

$$x_{ij} = \begin{cases} 1 & \text{if factory } i \text{ is constructed in year } j \\ 0 & \text{if factory } i \text{ is not constructed in year } j \end{cases}$$

and c_{ij} = cost of building factory i in year j.

Set up the integer programming problem to minimise the total costs over the five year period provided all the factories are built.

For each of the **separate** cases below, list the additional constraints required.

(a) Factory 2 cannot be built before factory 6, but they can be built in the same year.

(b) Factories 1, 3, 7 must be built by the end of year 4.

(c) If factories 3 and 4 are built in a given year, then no other factories can be built in that year.

(d) In the first three years either factories 3, 4 or factories 5, 6 must be built.

(12 marks)

 Question 1 continued on next page

1 (continued)

(ii) Alec travels frequently by car between two cities and has two route options: route A is a fast highway and the travel cost is £5; route B is a long winding road and the travel cost is £20. The full police force can be allocated to either route A or route B. Alec, with his desire for driving fast, is certain to receive a £100 speeding ticket on either route if the police are on that route. Alternatively, if half the police force is allocated to route A and half the police force to route B, there is a 50% chance he will receive a £100 ticket on route A and only a 30% chance he will receive a £100 ticket on route B.

The police force has three options; namely, the full police force on route A, a 50-50 split of the force on routes A and B and the full police force on route B.

Determine the payoff matrix with Alec as the "row player".

Determine the optimal mixed strategy for Alec using the two-variable graphical approach.

Establish which police strategy is not included in the optimal mixed strategy solution. Hence, determine the optimal mixed strategy for the police. ***(13 marks)***

2 (i) Prove that, if the **PRIMAL** maximising Linear Programming problem has an unbounded feasible optimal solution, then the associated **DUAL** problem has no feasible solution. ***(6 marks)***

2 (continued)

(ii) Saturn are manufacturers of two types of mountain bicycles. The first is called an Eagle and the second is called the Falcon. Suppose that x_1 Eagle bikes and x_2 Falcon bicycles are produced. Saturn has priced the Eagle at £800 per bike and the Falcon at £1250 per bike. The company wishes to maximise its revenue. The Linear Programming Problem model to maximise the revenue is

$$\max \quad 800x_1 + 1250x_2$$

subject to the constraints

$$\begin{aligned} x_1 + 2x_2 &\leq 20 && \text{welding constraint} \\ 500x_1 + 750x_2 &\leq 9000 && \text{brake component constraint} \\ -2x_1 + 3x_2 &\leq 0 && \text{market research constraint} \end{aligned}$$

From the tableaux below, with variables x_3, x_4 and x_5 the respective slack variables, write down the optimal solution and indicate which constraints are binding. ***(4 marks)***

Part of the Initial Tableau showing Constraints

	x_1	x_2	x_3	x_4	x_5	Solution
x_3	1	2	1	0	0	20
x_4	500	750	0	1	0	9000
x_5	-2	3	0	0	1	0

Optimal Tableau

	x_1	x_2	x_3	x_4	x_5	Solution
z	0	0	100	1.4	0	14600
x_1	1	0	-3	0.008	0	12
x_5	0	0	-12	0.028	1	12
x_2	0	1	2	-0.004	0	4

(a) Determine the range of values of the selling price of Falcons for which the current basis remains optimal. ***(6 marks)***

(b) Determine the range of values of the welding constraint right hand side value for which the current basis remains optimal. ***(5 marks)***

(c) Determine the change in the value of the revenue if one unit is added to the right hand side of the brake component constraint (assuming that the final basis remains feasible). ***(4 marks)***

3 (i) With reference to the following linear programming problem:

$$\text{maximise} \quad z = x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 10 \\ 2x_1 + 3x_2 &\geq 10 \\ 3x_1 + 2x_2 &= 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

give reasons for the need for **artificial** variables and indicate how the Simplex method can be modified to take account of them. ***(4 marks)***

(ii) Use the Two-Phase Simplex method to solve the problem. ***(21 marks)***

4 (i) Using the usual notation, the tableau for the problem

$$\text{maximise} \quad z = \mathbf{c}_1^T \mathbf{x}_1 + \mathbf{c}_2^T \mathbf{x}_2$$

subject to

$$\begin{aligned} A\mathbf{x}_1 + I\mathbf{x}_2 &= \mathbf{b} \\ \mathbf{x}_1, \mathbf{x}_2 &\geq \mathbf{0}, \end{aligned}$$

at a stage in the Simplex algorithm when the basis matrix is B, is represented by

$$\begin{bmatrix} 1 & \mathbf{c}_B^T B^{-1} A - \mathbf{c}_1^T & \mathbf{c}_B^T B^{-1} - \mathbf{c}_2^T \\ \mathbf{0} & B^{-1} A & B^{-1} \end{bmatrix} \begin{bmatrix} z \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B^T B^{-1} \mathbf{b} \\ B^{-1} \mathbf{b} \end{bmatrix}$$

where $\mathbf{x}_2$ is the starting basis.

(a) Write down the conditions necessary for this tableau to be optimal. ***(3 marks)***

(b) Write down an expression for the optimal dual variables in terms of the notation above and verify that the dual constraints are satisfied. ***(7 marks)***

(ii) In the context of deriving the dual of a maximisation linear programming problem, describe the transformations necessary to convert the following into the Standard Form:

(a) $4x_1 + 3x_2 = 10$

(b) x_3 unrestricted

(c) minimise $z = 10x_1 + 3x_2$

(d) $2x_1 - x_2 \geq 4$ ***(4 marks)***

Hence, from **first principles**, derive the dual of the linear programming problem

$$\text{minimise} \quad v = 4y_1 - y_2$$

subject to

$$\begin{aligned} y_1 + 2y_2 &= 4 \\ 2y_1 + 2y_2 &\geq 3 \\ y_1 + y_2 &\leq 8 \\ y_1 &\quad \text{unrestricted} \\ y_2 &\geq 0 \end{aligned}$$

(11 marks)

5 The optimal tableau for the problem

$$\text{maximise} \quad z = x_1 + x_2 + 3x_3$$

subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 &\le 12 & (1)\\ x_1 - x_2 + x_3 &\le 8 & (2)\\ x_1 + 2x_2 - x_3 &\le 6 & (3)\\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

is

	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	2	1	0	32
x_2	1/2	1	0	1/2	−1/2	0	2
x_3	3/2	0	1	1/2	1/2	0	10
x_6	3/2	0	0	−1/2	3/2	1	12

where x_4, x_5 and x_6 are the slack variables associated with constraints (1), (2) and (3) respectively.

(i) Write down the values of the dual variables and, using the inverse of the basis matrix, verify their values. ***(5 marks)***

(ii) Verify, by adding them to the optimal tableau **separately**, whether or not the addition of the constraints

(a) $2x_1 + x_2 \le 4$

(b) $x_1 + x_2 + x_3 \le 10$

changes the optimal solution and, if so, determine the new solution. ***(14 marks)***

(iii) A new variable is added to the problem which has cost coefficient 4 and coefficients 1, 1 and 2 in constraints (1), (2) and (3), respectively. Indicate whether the addition of this new variable alters the solution of the problem, giving reasons for your answer. If the answer is YES, then indicate **briefly** what changes to the optimal tableau would be necessary before the Simplex algorithm could be applied to determine the new solution. DO NOT SOLVE THE RESULTING PROBLEM. ***(6 marks)***

End of Question Paper