The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011-2012**

Milestones in Applied Mathematics II **2 Hours**

Marks will be awared for your best FOUR answers.

1 (i) An ultraviolet light source has power 10^{-2} W and emits light of wavelength 10^{-7} m. An aluminium plate is situated close to the source. The photoelectric work function for aluminium is 6.74×10^{-19} J, and the mass of an electron is 9.11×10^{-31} kg. Using the value $h = 6.626 \times 10^{-34}$ J s for Planck's constant and the value 3×10^{8} m s^{-1} for the speed of light, calculate the following, giving each answer correct to 3 significant figures and clearly stating the units:

(a) The energy of each photon of light emitted by the source.

(b) The maximum kinetic energy of electrons emitted by the aluminium when the ultraviolet light from the source is incident on the aluminium plate.

(c) The velocity and de Broglie wavelength of the electrons emitted with maximum kinetic energy. ***(13 marks)***

(ii) The wave-function of a free particle of mass m is given by

$$\psi(x,t) = Ae^{i(kx-\omega t)},$$

where A is a complex constant, and k and ω are real constants.

Find the energy and momentum of the particle, and show that $\omega = \dfrac{\hbar k^2}{2m}$.
Find the probability current for this wave-function. ***(12 marks)***

2 Consider a particle of mass m moving in a time-independent potential $V(\mathbf{x})$. Assume that a state of the particle is described by a wavefunction of the form $\Psi(\mathbf{x}, t) = \phi(\mathbf{x})\chi(t)$.

(i) Show that $\chi(t) = Ae^{-iBt/\hbar}$ (A and B are constant) and that $\phi(\mathbf{x})$ satisfies

$$-\frac{\hbar^2}{2m}\nabla^2\phi(\mathbf{x}) + V(\mathbf{x})\phi(\mathbf{x}) = B\phi(\mathbf{x}).$$

(13 marks)

(ii) Consider the following wavefunction:

$$\Psi(x, t) = Ce^{\frac{Dx^2}{2}}e^{-i\omega t}.$$

with C, D and ω are **real** constants and $D > 0$. Find the potential $V(x)$ for which this wavefunction is the solution of Schrödinger's equation. Why is this wavefunction not a realistic possible wavefunction? ***(12 marks)***

3 The wave function $\Psi(x)$ satisfies the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + V(x)\Psi = E\Psi,$$

where the potential is such that

$$\frac{2m}{\hbar^2}[E - V(x)] = \begin{cases} 4\pi^2 & x < 0; \\ 16\pi^2 & x > 0. \end{cases}$$

(i) State the boundary conditions that must be satisfied by $\Psi(x)$ at $x = 0$.

Particles of energy E are incident from minus infinity on this potential. Show that

$$\Psi(x) = \begin{cases} Ae^{2\pi ix} + Be^{-2\pi ix}, & x < 0; \\ Ce^{4\pi ix}, & x > 0, \end{cases}$$

where A, B and C are complex constants. ***(10 marks)***

(ii) Use the boundary conditions at $x = 0$ to find A and B in terms of C. Define the probability current $j(x)$, and show that, for $x > 0$,

$$j(x) = 4\pi\hbar|C|^2.$$

Hence show that the transmission coefficient T is equal to 8/9.

(15 marks)

4 The Hamiltonian of a quantum system is given by

$$H = \hbar\omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the energy eigenvalues of the system and the corresponding normalized eigenvectors. ***(10 marks)***

At time $t = 0$, the state of the system is given by

$$\Psi(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Show that, at a later time t, the state of the system is given by

$$\Psi(t) = \frac{1}{2}e^{-i\omega t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}e^{i\omega t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Show that the probability that, at time t, the system is observed to be in the state

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is $\cos^2(\omega t)$. ***(15 marks)***

5 The Hamiltonian of a quantum system is

$$H = \frac{1}{2}P^2 + \frac{1}{2}X^2,$$

where P is the momentum operator and X is the position operator. You are given that the commutator of X and P is

$$[X, P] = i\hbar.$$

(i) Show that

(a) $[H, P] = i\hbar X$,

(b) $[H, X] = -i\hbar P$,

(c) $[H, XP] = -i\hbar P^2 + i\hbar X^2$. ***(13 marks)***

(ii) Now consider a state ψ such that

$$H\psi = \frac{1}{2}\hbar\psi.$$

For the remainder of this question, you may use the fact that the expectation value of $[H, A]$ in the state ψ is zero for any operator A. Show that

(a) $E_\psi(P) = 0$; $E_\psi(X) = 0$.

(b) $E_\psi(P^2) = E_\psi(X^2) = \hbar/2$. ***(12 marks)***

End of Question Paper