The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011–2012**

Mathematical Methods **2 hours**

Marks will be awarded for your best ***FOUR*** *answers. The marks awarded to each question or section of question are shown in italics.*

1 The Fourier transform, $\hat{f}(k)$, of a function $f(x)$ is defined by

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx.$$

(a) Write down the inverse Fourier transform, and use it to derive Parseval's theorem:

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\hat{f}(k)\right|^2 \, dk.$$

(9 marks)

(b) The function $f(x)$ is defined by

$$f(x) = \begin{cases} 1 - x^2 & |x| \leqslant 1 \\ 0 & |x| > 1. \end{cases}$$

Find $\hat{f}(k)$, and use Parseval's theorem to deduce that

$$\int_0^{\infty} \left(\frac{\sin k - k\cos k}{k^3} \right)^2 dk = \frac{\pi}{15}\,.$$

(16 marks)

2 The Laplace transform of a function $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

(a) By using the change of variables $t^{1/2} = u$, show that for $\operatorname{Re} s > 0$

$$\mathcal{L}\left\{t^{-1/2}\right\} = \sqrt{\pi}\, s^{-1/2}.$$

(3 marks)

$$\left[\ \text{You may assume that } \int_0^\infty e^{-su^2}\, du = \frac{1}{2}\sqrt{\frac{\pi}{s}} \ \text{ for } \operatorname{Re} s > 0.\ \right]$$

Hence, by integrating by parts, show that

$$\mathcal{L}\left\{t^{n-1/2}\right\} = \sqrt{\pi}\,\frac{1}{2}\cdot\frac{3}{2}\cdots\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right) s^{-\left(n+\frac{1}{2}\right)},$$

where n is a positive integer and $\operatorname{Re} s > 0$. ***(7 marks)***

(b) Find the Laplace transform of $\sin \omega t$ for $\operatorname{Re} s > 0$. ***(4 marks)***

(c) If $F(t)$ is defined for $t > 0$ by

$$F(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau,$$

show that

$$\mathcal{L}\{F(t)\} = \mathcal{L}\{f(t)\}\, \mathcal{L}\{g(t)\}.$$

(5 marks)

(d) Use the results of parts (a), (b) and (c) to show that the inverse Laplace transform of

$$\frac{s^{-5/2}}{s^2+1}$$

is

$$\frac{4}{3\sqrt{\pi}} \int_0^t \tau^{3/2} \sin(t-\tau)\, d\tau.$$

(6 marks)

3 The function $y(x)$ satisfies the ordinary differential equation

$$y'' - 5y' + 4y = f(x)$$

in $0 \leqslant x \leqslant 1$, for some function $f(x)$, and also satisfies the boundary conditions

$$y(0) = y'(0) = 0.$$

(a) Find the independent solutions of

$$y'' - 5y' + 4y = 0.$$ ***(3 marks)***

(b) Given that Green's function, $G(x;\xi)$, for the boundary-value problem given at the beginning of the question is continuous at $x = \xi$, and that $\partial G/\partial x$ has a discontinuity of size 1 at $x = \xi$, find $G(x;\xi)$ and use it to show that the solution to the given boundary-value problem is

$$y(x) = \frac{1}{3}\int_0^x \left\{e^{4(x-\xi)} - e^{x-\xi}\right\} f(\xi)\, d\xi.$$ ***(11 marks)***

Hence find $y(x)$ when $f(x) = 4x$, and verify that it satisfies the differential equation and the boundary conditions. ***(11 marks)***

4 Consider the equation

$$(1-\epsilon)x^2 + 2x + 1 = 0, \qquad (*)$$

where ϵ is a constant satisfying $0 < \epsilon \ll 1$.

(a) The solution to equation $(*)$ can be written as

$$x = x_0 + \epsilon^{1/2}x_1 + \epsilon x_2 + \epsilon^{3/2}x_3 + \epsilon^2 x_4 + \epsilon^{5/2}x_5 + \cdots,$$

where $x_0, x_1, x_2, \ldots$ are $O(1)$ as $\epsilon \to 0$.

Use this expression to derive the two solutions to equation $(*)$, correct to $O(\epsilon^2)$ as $\epsilon \to 0$. ***(19 marks)***

(b) Find the exact solutions of $(*)$, and show that their expansions agree with your results from part (a). ***(6 marks)***

5 For real x, one of the modified Bessel functions, $K_n(x)$, is defined by

$$K_n(x) = \int_0^\infty e^{-x\cosh t}\cosh(nt)\,dt.$$

(a) Given that

$$\cosh t = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\,,$$

and defining

$$v = \cosh t - 1,$$

show that

$$t \sim \sqrt{2v}\left(1 - \frac{1}{12}v\right) \qquad \text{as } t, v \to 0.$$ ***(6 marks)***

Hence show that

$$K_n(x) \sim \frac{e^{-x}}{\sqrt{2}}\int_0^\infty v^{-1/2}e^{-xv}\,dv \qquad \text{as } x \to \infty.$$ ***(3 marks)***

By changing variable to $u = (xv)^{1/2}$, show that

$$K_n(x) \sim \sqrt{\frac{\pi}{2x}}\,e^{-x} \qquad \text{as } x \to \infty.$$ ***(5 marks)***

$$\left[\text{ You may assume that } \qquad \int_0^\infty e^{-u^2}\,du = \frac{1}{2}\sqrt{\pi}.\ \right]$$

(b) $K_n(x)$ satisfies the differential equation

$$K_n'' + \frac{1}{x}K_n' - \left(1 + \frac{n^2}{x^2}\right)K_n = 0. \qquad (\dagger)$$

Assume that, as $x \to \infty$, the asymptotic expansion of $K_n(x)$ is given by

$$K_n(x) \sim \sqrt{\frac{\pi}{2x}}\,e^{-x}\left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots\right),$$

for some constants $a_1, a_2, \ldots$

By substituting this expansion into the differential equation (†), and equating coefficients, find $K_n(x)$ correct to $O\left(x^{-3/2}\,e^{-x}\right)$ as $x \to \infty$.
(11 marks)

End of Question Paper