The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2011–2012

Metric spaces — 2 hours and 30 minutes

Answer four *questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f\colon X \to Y$ be a function.

(a) Define, in terms of convergent sequences, what it means for f to be *continuous*. (2 marks)

(b) Let $A \subset Y$. What is meant by $f^{-1}A$? What does it mean for A to be *closed*? Prove that if f is continuous and $A \subset Y$ is closed, then $f^{-1}A$ is closed. (12 marks)

(ii) Let $\mathbb{R}$ and $\mathbb{R}^2$ be given their usual Euclidean metrics, and let $g\colon \mathbb{R} \to \mathbb{R}$ be continuous.

(a) Show that the function $G\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$G(x, y) = y - g(x)$$

is also continuous. (7 marks)

(b) Deduce, using part (i)(b), that

$$\left\{(x, y) \in \mathbb{R}^2 \mid y \geq g(x)\right\}$$

is a closed subset of $\mathbb{R}^2$. (4 marks)

2 (i) (a) Define the *supremum* or d_∞ metric on $C[0,1]$. (2 marks)

(b) Define what it means for a sequence (f_n) in $C[0,1]$ to *converge pointwise* to a function $f \in C[0,1]$. (2 marks)

(c) Prove that if (f_n) converges to f in the metric space $(C[0,1], d_\infty)$, then (f_n) converges to f pointwise. (6 marks)

(ii) Which of the following sequences converges pointwise to the zero function $g\colon [0,1] \to \mathbb{R}$ defined by $g(x) = 0$ for all $x \in [0,1]$? In each case either prove the convergence, or explain why convergence fails.

(a) The sequence (h_n) with $h_n(x) = \dfrac{x}{n}$.

(b) The sequence (k_n) with $k_n(x) = x^n$.

(6 marks)

(iii) Consider the sequence (f_n) in $C[0,1]$ defined by

$$f_n(x) = \begin{cases} nx & \text{if } 0 \le x \le 1/n \\ n\left(\dfrac{2}{n} - x\right) & \text{if } 1/n \le x \le 2/n \\ 0 & \text{if } 2/n \le x \end{cases}$$

(a) Sketch the graph of f_n.

(b) Prove that (f_n) converges pointwise to the zero function g.

[*Hint: You should use the fact that for* $x \neq 0$, $f_n(x) = 0$ *for* $n \geq 2/x$.]

(c) Show that (f_n) does not converge to the zero function g in $(C[0,1], d_\infty)$.

(9 marks)

3 (i) (a) Define what it means for $d: X \times X \to \mathbb{R}$ to be a *metric* on a set X. (4 marks)

(b) Recall that the *Euclidean* or d_2 metric on $\mathbb{R}^n$ is defined by

$$d_2\Big((x_1, \dots, x_n), (y_1, \dots, y_n)\Big) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}.$$

Prove that d_2 is indeed a metric.

[*You may assume the* Cauchy-Schwarz inequality*:*

$$\sum a_i b_i \leq \left(\sum a_i^2\right)^{\frac{1}{2}} \left(\sum b_i^2\right)^{\frac{1}{2}}$$

for $a_1, \dots, a_n$, $b_1, \dots, b_n$ *real numbers.*] (10 marks)

(ii) (a) Now consider $\mathbb{R}^2$ with the Euclidean metric d_2. Show that $(x_n, y_n) \to (x, y)$ if and only if $x_n \to x$ and $y_n \to y$. (6 marks)

(b) Show that $\left(\sin\left(n^{-\frac{3}{2}}\right), \ln\left(\frac{n+1}{n}\right)\right) \to (0, 0)$ in the metric space $(\mathbb{R}^2, d_2)$. (5 marks)

4 (i) Let (X, d) be a metric space and let $f: X \to X$ be a function.

(a) What does it mean for f to be a *contraction*? What is meant by a *fixed point* of f? (3 marks)

(b) Let x_1 be any point of X and define a sequence (x_n) iteratively by setting $x_{n+1} = f(x_n)$ for all $n \geq 1$. Suppose that (x_n) converges to some $x \in X$. Show that if f is continuous then x is a fixed point of f. (4 marks)

(ii) Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = \left(\frac{3}{2} - \frac{y}{2}, \frac{1}{2} + \frac{x}{2}\right)$$

(a) Show that f is a contraction of the metric space $(\mathbb{R}^2, d_2)$. (9 marks)

(b) Set $(x_1, y_1) = (0, 0)$. Compute the terms $(x_2, y_2), \dots, (x_5, y_5)$ in the sequence $((x_n, y_n))$ determined by iterating $(x_{n+1}, y_{n+1}) = f(x_n, y_n)$. Guess a value for the limit of the sequence $((x_n, y_n))$. (3 marks)

(c) What are the fixed points of f? In your answer you should:

- Check that the points you list really are fixed points.
- Explain why there are no other fixed points than the ones you have listed.

(6 marks)

5 (i) (a) What does it mean for a metric space (X, d) to be *compact*? (2 marks)

(b) What is a *complete* metric space? (2 marks)

(c) Show that a compact metric space (X, d) is complete. (7 marks)

(ii) Consider the sequence (f_n) in $C[0, 1]$, where the f_n are defined by

$$f_n(x) = \begin{cases} 2^n x & \text{if } x \leq 1/2^n \\ 1 & \text{if } x \geq 1/2^n \end{cases}$$

(a) Sketch the graph of f_n. (2 marks)

(b) Show that $d_\infty(f_n, f_m) \geq 1/2$ when $n \neq m$.
[*Hint: Assume* $n > m$ *and calculate* $f_n(1/2^n)$, $f_m(1/2^n)$.] (6 marks)

(c) Show that (f_n) has no subsequence which is convergent in the d_∞ metric. (4 marks)

(d) Show that $(C[0, 1], d_\infty)$ is not compact. (2 marks)

End of Question Paper