The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011-2012**

Complex Analysis **2 hours 30 minutes**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Express both of the following in the form $x + iy$:

$$\frac{18-i}{3+4i}\ ; \qquad (1+i)^9\ .$$ ***(4 marks)***

(ii) Express

$$\frac{(1+i)^{11}}{(1+i\sqrt{3})^5}$$

in the form $re^{i\theta}$ with $r > 0$ and $-\pi < \theta \leq \pi$. ***(4 marks)***

(iii) State, without proof, the triangle inequalities for $|\,z+w|$ and $|\,z-w|$.

Let $S = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 4 \quad \text{and} \quad 1 \leq \operatorname{Im} z \leq 3\}$. Sketch S and show that, for all points $z \in S$,

$$\frac{1}{12e^3} \leq \left| \frac{e^{iz}}{z+7} \right| \leq \frac{1}{2e}\ .$$ ***(7 marks)***

(iv) Find all the solutions of the equation $z^4 + 1 = 0$.
Hence express $x^4 + 1$ as the product of two real quadratic factors. ***(5 marks)***

(v) Find all the solutions of the equation

$$\sin z = 2i.$$ ***(5 marks)***

2 (i) Define what is meant by the following two statements:

(a) A function f is **differentiable at the point** z_0 ;

(b) A function f is **analytic in a region** D. ***(2 marks)***

Let

$$g(z) = \frac{\sin(\pi z)}{(1 + e^{\pi i z})^5} \, .$$

Decide where g is analytic giving reasons for your answer. ***(5 marks)***

(ii) State, without proof, the Cauchy-Riemann equations for a differentiable function. ***(1 mark)***

Let

$$h(z) \;=\; |z|^2 \, .$$

Prove that, if $z_0 \;\neq\; 0$, then the function h is not differentiable at z_0. Is h differentiable at the origin? Give reasons for your answer. ***(6 marks)***

(iii) In each of the following cases, determine whether there is a function k analytic on $\mathbb{C}$ with $\mathrm{Re}\,(k(x+iy)) = u(x,y)$, giving reasons for your answers:

(a) $u(x,y) \;=\; x^2 + xy + 1 \, ,$

(b) $u(x,y) \;=\; y + \sinh x \cos y \, .$

When k exists, find an explicit expression for $k(z)$ in terms of z and show that you have all functions satisfying the conditions. ***(7 marks)***

(iv) Let α be the semi-circular path given by $z \;=\; 2e^{it} \quad (0 \leq t \leq \pi)$. Show that

$$\left| \int_\alpha \frac{\sin z + \overline{z}}{z^2} \, dz \right| \;\leq\; \frac{\pi(2 + \cosh 2)}{2} \, .$$

(4 marks)

3 State, without proof, Cauchy's Theorem and Cauchy's Integral Formulae for a function and for its derivatives. Your statement should include conditions under which the results are valid. ***(7 marks)***

The contour γ consists of the semi-circular path given by $z = 4e^{it}$ $(0 \leq t \leq \pi)$ followed by the straight line segment from -4 to 4. Sketch γ.

Without using the Residue Theorem, evaluate

(i) $\displaystyle\int_\gamma \frac{\sin(\pi z^2)}{z-i}\,dz$, (ii) $\displaystyle\int_\gamma \frac{z\,e^z}{(z^2+25)^2}\,dz$,

(iii) $\displaystyle\int_\gamma \frac{z^2\cosh(\pi z)}{z^2+4}\,dz$, (iv) $\displaystyle\int_\gamma \overline{z}\,dz$,

(v) $\displaystyle\int_\gamma \frac{e^z}{(2z-i\pi)^{10}}\,dz$.

(18 marks)

4 (i) Let S denote the strip $S = \{z \in \mathbb{C} : -1 < \text{Re}z < 1\}$ in the complex plane. Show that for **all paths** α in S from $-i$ to i,

$$\int_\alpha \frac{dz}{1-z^2} \quad = \quad \frac{i\pi}{2}\ .$$ ***(5 marks)***

(ii) Find the radius of convergence of the following power series :

$$\sum_{n=0}^{\infty}(4+3i)^n z^n\ .$$ ***(2 marks)***

(iii) Describe how Laurent expansions are used to define the classification of isolated singularities. ***(6 marks)***

Find **all** the singularities in $\mathbb{C}$ of both of the following functions:

$$(z+1)\sin\left(\frac{1}{z^2}\right)\ , \qquad\qquad \frac{z+2}{\sin(\pi z)}\ .$$

Classify these singularities, giving reasons for your answers and evaluate the residue at each of them. ***(12 marks)***

5 (i) State, without proof, Cauchy's Residue Theorem. Your statement should include conditions under which the result is valid. ***(4 marks)***

Let γ be the circular contour $|z| = 3$ described in the anti-clockwise direction. Evaluate

$$\int_\gamma \frac{(z+1)\,\exp\left(\frac{1}{z}\right)}{z}\,dz\,, \qquad \int_\gamma \frac{1}{e^{\pi z}-1}\,dz\,.$$

(11 marks)

(ii) Prove that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+2}\,dx = -\frac{\pi}{e}\,\sin 1\,.$$

(10 marks)

End of Question Paper