



Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.

- 1 (i) State the Subfield Criterion. (4 marks)
- (ii) For each of the subsets J_1, J_2 of \mathbb{C} specified below determine, with justification, whether it is a subfield of \mathbb{C} :
- (a) $J_1 = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$, (5 marks)
- (b) $J_2 = \{a + bi\sqrt{2} + ci : a, b, c \in \mathbb{Q}\}$. (3 marks)
- (iii) Consider the subfield $L = \mathbb{Q}(\sqrt{2}, i\sqrt{2})$ of \mathbb{C} .
- (a) Find $[L : \mathbb{Q}]$. Justify your answer and give a \mathbb{Q} -basis of L . (7 marks)
- (b) Prove that $L = \mathbb{Q}(i, \frac{1}{\sqrt{2}})$. (3 marks)
- (c) Find $\frac{1 + i\sqrt{2}}{1 - i\sqrt{2}}$. The answer should be given in terms of the basis of (a). (3 marks)
- 2 (i) Let K be a subfield of a field L . Give a definition of $[L : K]$. (2 marks)
- (ii) State the degrees formula for finite field extensions $K \subseteq L \subseteq M$. (2 marks)
- (iii) Prove the degrees formula for finite field extensions $K \subseteq L \subseteq M$. (9 marks)
- (iv) Let $K_0 \subseteq K_1 \subseteq \dots \subseteq K_n$ be finite field extensions. Prove that
$$[K_n : K_0] = [K_n : K_{n-1}][K_{n-1} : K_{n-2}] \cdots [K_1 : K_0].$$
 (5 marks)
- (v) Using (iv) (or otherwise) find $[\mathbb{Q}(\sqrt{2}, \sqrt{3}, i) : \mathbb{Q}]$. (7 marks)

- 3 (i) (a) State Eisenstein's Irreducibility Criterion. (2 marks)
- (b) Prove Eisenstein's Irreducibility Criterion. (8 marks)
- (c) Show that the polynomial $x^5 - 15x^4 - 20x^3 - 35x^2 + 10$ is irreducible in $\mathbb{Q}[x]$. (2 marks)
- (ii) (a) State the "shifted version" of Eisenstein's Irreducibility Criterion. (3 marks)
- (b) Prove the "shifted version" of Eisenstein's Irreducibility Criterion. (5 marks)
- (c) Show that the polynomial $f(x) = 2x^3 - 3x^2 + 4$ is irreducible in $\mathbb{Q}[x]$ by using the "shifted version" of Eisenstein's Irreducibility Criterion (or otherwise). (5 marks)
- 4 (i) (a) Give the definition of a simple field extension. (2 marks)
- (b) Give an example of a simple field extension. (2 marks)
- (c) Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Prove that $K = \mathbb{Q}(b)$ where $b = \sqrt{2} + \sqrt{3}$. (8 marks)
- (ii) (a) Give the definition of the n -th cyclotomic field. Give a complex number which generates it. (3 marks)
- (b) Let p be a prime number. Write down the p -th cyclotomic polynomial as a polynomial with integer coefficients, and also as a product of linear factors in $\mathbb{C}[X]$. (2 marks)
- (c) Let p be a prime number. Prove that the p -th cyclotomic polynomial is irreducible in $\mathbb{Q}[X]$. (8 marks)

- 5 (i) (a) Specify the four standard constructions that are used in the theory of ruler-and-compass constructions and involve perpendicular and parallel lines. **(7 marks)**
- (b) Give the definition of a constructible real number. **(2 marks)**
- (c) Let $a, b \in \mathbb{R}$ be constructible numbers. Using the four standard constructions prove that the numbers
- $$a - b \text{ and } a + b$$
- are constructible (You may use the fact that a point (x, y) is constructible if and only if the point (y, x) is constructible if and only if the numbers x and y are constructible). **(6 marks)**
- (d) Show that the number $1 + \sqrt{3 + \sqrt[4]{1 + \sqrt[4]{2}}}$ is constructible (You may use the fact that if x is constructible then so is \sqrt{x}). **(4 marks)**
- (ii) Let $a \in \mathbb{R}$. Prove that a is a constructible real number if and only if $(0, a)$ is a constructible point in the plane. **(6 marks)**

End of Question Paper