The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–12**

Combinatorics **2 hours 30 minutes**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

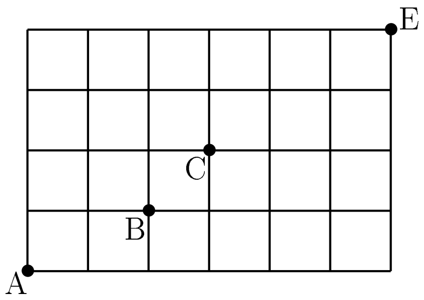
1 (i) Show, by each of the following two methods, that

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

(a) Count the ways of choosing a (non-empty) subset of a set of n people, with one of the subset designated as leader. ***(5 marks)***

(b) Differentiate both sides of the binomial identity:
$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$ ***(4 marks)***

(ii) This part of the question concerns routes in the grid illustrated:



(a) How many shortest routes are there from A to E along the lines of the grid? Give a brief reason for your answer. ***(3 marks)***

(b) Find the number of such routes which go through B and the number of such routes which go through B and C. ***(4 marks)***

(iii) Consider the equation

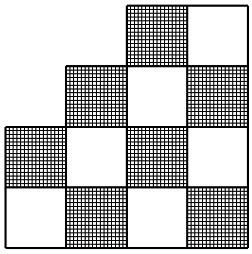
$$x_1 + x_2 + \cdots + x_k = n.$$

(a) How many solutions are there of this equation in which each x_i is a non-negative integer? Give a brief reason for your answer.
(3 marks)

(b) How many of the solutions have each x_i equal to 0 or 1?
(3 marks)

(c) For a positive integer r, how many of the solutions have x_r as the first positive number in the list $x_1, x_2, \ldots, x_k$? ***(3 marks)***

2 (i) Consider an $n \times n$ chess board with three squares removed from one corner - the corner square and the two adjacent edge squares. (The 4×4 case is indicated in the diagram below.) Show that the board cannot be completely covered by non-overlapping dominoes (that is, by pieces which cover exactly two adjacent squares).



(7 marks)

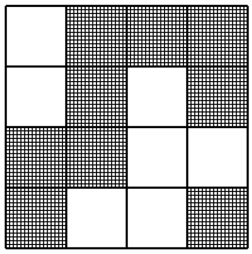
(ii) (a) State the Pigeon-hole Principle. ***(2 marks)***

(b) Prove that there are two different powers of 3 whose difference is divisible by 1000. ***(4 marks)***

(iii) (a) State the Inclusion/Exclusion Principle. ***(3 marks)***

(b) How many integers from 1 to 8000 are divisible by at least one of 2, 3 and 7 ? ***(9 marks)***

3 (i) (a) Calculate the rook polynomial of the (unshaded) board B:



(7 marks)

(b) Use your answer to calculate the number of ways of placing 4 non-challenging rooks on the complement of B (i.e. the *shaded* board).
(3 marks)

(ii) Which of the following polynomials can be the rook polynomial of a board? Give reasons for your answers, including examples of appropriate boards.

(a) $1 - 6x$.

(b) $1 + \frac{3}{4}x$.

(c) $(1 + x)(1 + 4x + 2x^2)$.

(d) $1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$.

(e) $1 + \binom{n}{1}(3x) + \binom{n}{2}(3x)^2 + \cdots + \binom{n}{n}(3x)^n$. ***(8 marks)***

(iii) Jobs a, b, c, d are to be allocated to people A, B, C, D with each person getting one of the jobs. Their unsuitability for the jobs is shown in the following table, lower numbers meaning that people are more suitable for jobs. Use the Hungarian algorithm to allocate the jobs with lowest total unsuitability.

	a	b	c	d
A	4	0	2	3
B	3	1	0	0
C	0	1	2	3
D	4	0	3	3

(7 marks)

4 (i) State necessary and sufficient conditions for a $p \times q$ Latin rectangle to be extendable to an $n \times n$ Latin square. ***(2 marks)***

(ii) For what value of x can the following Latin rectangle be extended to a 6×6 Latin square?

$$\begin{pmatrix} 1 & 4 & 6 \\ 3 & 2 & 1 \\ 6 & 1 & 3 \\ 4 & x & 2 \end{pmatrix}$$

Write down one such extension. ***(7 marks)***

(iii) Now let L be a $p \times p$ Latin rectangle, where $p < n$, with entries in $\{1, 2, \ldots, n\}$. Suppose that 1 occurs in each row of L and that the other numbers $2, 3, \ldots, n$ all occur in L the same number of times. Show that L can be extended to an $n \times n$ Latin square. ***(8 marks)***

(iv) Prove that there exist at most $n - 1$ mutually orthogonal $n \times n$ Latin squares. ***(8 marks)***

5 (i) Which of the following are possible scores in a tournament of 7 people?

(a) $6, 5, 5, 3, 2, 1, 0$.

(b) $6, 5, 5, 2, 2, 1, 0$.

(c) $5, 5, 5, 3, 2, 1, 0$. ***(7 marks)***

(ii) Let $w_1, w_2, \ldots, w_n$ be the scores in a tournament. Show that the players can be divided into two non-empty subsets W and L such that each player in W beats each player in L if and only if there exists an integer r with $1 \leq r < n$ such that some r of the scores add to precisely $\binom{r}{2}$. ***(6 marks)***

(iii) Let $p = 4m + 3$ be a prime number. Explain (without proof) how to use quadratic residues modulo p to construct a $(4m+3, 4m+3, 2m+1, 2m+1, m)$ design. ***(4 marks)***

(iv) Consider a (v, b, r, k, λ) design. Give two equations expressing r in terms of the other parameters of the design. ***(3 marks)***

(v) Do there exist designs with the following parameters? Give brief reasons for your answers.

(a) $(11, 11, 5, 5, 2)$.

(b) $(11, 11, 4, 6, 2)$.

(c) $(11, 11, 6, 6, 2)$. ***(5 marks)***

End of Question Paper