The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–12**

Differential Geometry **2 hours 30 minutes**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

A list of formulae is provided on the last page.

1 Let $\mathbf{x}(s)$ be a unit-speed curve in $\mathbb{R}^3$ defined for $s \in (-1, 1)$. The notations $\tau(s)$, $\mathbf{N}(s)$ and $\mathbf{B}(s)$ have the usual meaning of torsion, principal normal vector and binormal vector respectively. It is known that $\mathbf{x}(0) = (0, 0, 0)$, $\tau(0) > 0$, and

$$\mathbf{B}(s) = \left(-\frac{\sqrt{1+s}}{2}, \frac{\sqrt{1-s}}{2}, \frac{1}{\sqrt{2}} \right) \quad \text{for } s \in (-1, 1).$$

(i) Find an equation of the osculating plane at $s = 0$. ***(5 marks)***

(ii) Show that $\tau(s) = \dfrac{1}{2\sqrt{2(1-s^2)}}$ and $\mathbf{N}(s) = \left(\sqrt{\dfrac{1-s}{2}}, \sqrt{\dfrac{1+s}{2}}, 0 \right)$. ***(9 marks)***

(iii) Find the total angle of rotation of the osculating plane as s increases steadily from 0 to $1/2$. You may leave your answer as an integral. ***(3 marks)***

(iv) Find $\mathbf{x}(s)$ for $s \in (-1, 1)$. ***(8 marks)***

2 In this question, consider the cylinder

$$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$$

as well as curves of the form

$$\gamma_{a,b}(t) = (\cos at, \sin at, bt), \quad \text{where } a, b \in \mathbb{R}.$$

Notice that each $\gamma_{a,b}$ is a curve on C passing through the point $(1, 0, 0)$.

(i) Calculate the curvature of $\gamma_{a,b}$. ***(8 marks)***

(ii) Show that each $\gamma_{a,b}$ is a geodesic on C. ***(8 marks)***

(iii) Are there any other geodesics on C passing through $(1, 0, 0)$? No explanation is required. ***(3 marks)***

(iv) What is the value of b such that $\gamma_{1,b}$ contains the shortest path on C from $(1, 0, 0)$ to $(0, 1, \pi)$? Calculate the distance on C between the two points. ***(6 marks)***

3 Let S be the surface in $\mathbb{R}^3$ parametrized by

$$\mathbf{x}(u, v) = \left(u - \frac{u^3}{3} + uv^2\,,\ v - \frac{v^3}{3} + u^2 v\,,\ u^2 - v^2 \right).$$

For each $a \in \mathbb{R}$, let γ_a be the curve on S given by $\gamma_a(u) = \mathbf{x}(u, a)$.

(i) Show that the first fundamental form of $\mathbf{x}(u, v)$ is

$$(1 + u^2 + v^2)^2(du^2 + dv^2).$$ ***(8 marks)***

(ii) Calculate the area of the part of S defined by $u, v \in [0, 1]$. ***(8 marks)***

(iii) Write down a conformal map from $\mathbb{R}^2$ to S. ***(3 marks)***

(iv) Using your answer in (iii), write down the curve on S that passes through $\mathbf{x}(0, 0)$ and makes a right angle with every γ_a. Explain your answer. ***(6 marks)***

4 Let $R > r > 0$. The circle on the xz-plane with radius r and centred at $(R, 0, 0)$ is rotated about the z-axis to produce a torus. This torus can be parametrized by

$$\mathbf{x}(\phi, \theta) = \big((R + r\cos\phi)\cos\theta, (R + r\cos\phi)\sin\theta, r\sin\phi\big)$$

whose first fundamental form is $r^2 d\phi^2 + (R + r\cos\phi)^2 d\theta^2$.

(i) Show that the second fundamental form of $\mathbf{x}(\phi, \theta)$ is

$$r\,d\phi^2 + (R + r\cos\phi)\cos\phi\,d\theta^2.$$ ***(8 marks)***

(ii) Find the principal curvatures. ***(6 marks)***

(iii) Describe two curves on the torus passing through $(R, 0, r)$ with the property that their tangent vectors are always principal. ***(6 marks)***

(iv) Prove that the mean curvature of the torus is never zero if and only if $R > 2r$. ***(5 marks)***

5 The curve $x = \cosh z$ on the xz-plane is rotated about the z-axis to produce a surface of revolution (a catenoid). This surface S can be parametrized by

$$\mathbf{x}(t, \theta) = (\cosh t\cos\theta, \cosh t\sin\theta, t)$$

whose first and second fundamental forms are respectively

$$\cosh^2 t(dt^2 + d\theta^2) \quad \text{and} \quad -dt^2 + d\theta^2.$$

For $a \in \mathbb{R}$, let γ_a be the curve given by $\gamma_a(\theta) = \mathbf{x}(a, \theta)$. In other words, γ_a is the horizontal section of S at $z = a$.

(i) Calculate the Gaussian curvature K. ***(6 marks)***

(ii) If R is the region on S between γ_0 and γ_a, show that

$$\iint_R K\,dA = -2\pi\tanh a.$$

You may use the fact that $\dfrac{d}{dx}\tanh x = \operatorname{sech}^2 x$. ***(8 marks)***

(iii) The geodesic curvature κ_g of each γ_a is constant. In particular, $\kappa_g = 0$ for γ_0 because it is a geodesic. What is κ_g for γ_a in general? ***(6 marks)***

(iv) Can S have any simple closed geodesics? Explain why. ***(5 marks)***

End of Question Paper

Formula sheet

The following apply to either a unit-speed parametrization $\mathbf{x}(s)$ or a general (regular) parametrization $\mathbf{x}(t)$ of a smooth curve in $\mathbb{R}^3$:

- curvature and torsion

$$\kappa(t) = \frac{||\mathbf{x}'(t) \times \mathbf{x}''(t)||}{||\mathbf{x}'(t)||^3}, \qquad \tau(t) = \frac{[\mathbf{x}'(t) \times \mathbf{x}''(t)] \cdot \mathbf{x}'''(t)}{||\mathbf{x}'(t) \times \mathbf{x}''(t)||^2}$$

- Frenet frame

$$\mathbf{T}(t) \times \mathbf{N}(t) = \mathbf{B}(t), \qquad \mathbf{N}(t) \times \mathbf{B}(t) = \mathbf{T}(t), \qquad \mathbf{B}(t) \times \mathbf{T}(t) = \mathbf{N}(t)$$

- Frenet-Serret equations

$$\begin{aligned} \mathbf{T}'(s) &= \phantom{-\kappa(s)\mathbf{T}(s)} \quad \kappa(s)\mathbf{N}(s) \\ \mathbf{N}'(s) &= -\kappa(s)\mathbf{T}(s) \qquad\qquad + \tau(s)\mathbf{B}(s) \\ \mathbf{B}'(s) &= \qquad\qquad\quad - \tau(s)\mathbf{N}(s) \end{aligned}$$

The following apply to a (regular) parametrization $\mathbf{x}(u, v)$ of a smooth surface in $\mathbb{R}^3$:

- first fundamental form

$$E\,du^2 + 2F\,du\,dv + G\,dv^2, \qquad E = \mathbf{x}_u \cdot \mathbf{x}_u,\ \ F = \mathbf{x}_u \cdot \mathbf{x}_v,\ \ G = \mathbf{x}_v \cdot \mathbf{x}_v$$

- surface area

$$dA = \sqrt{EG - F^2}\,du\,dv$$

- unit normal vector

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{||\mathbf{x}_u \times \mathbf{x}_v||}$$

- second fundamental form

$$L\,du^2 + 2M\,du\,dv + N\,dv^2, \qquad L = \mathbf{x}_{uu} \cdot \mathbf{N},\ \ M = \mathbf{x}_{uv} \cdot \mathbf{N},\ \ N = \mathbf{x}_{vv} \cdot \mathbf{N}$$

- Weingarten matrix

$$\mathcal{W} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

- mean and Gaussian curvatures

$$H = \frac{1}{2}\,\mathrm{Tr}\,\mathcal{W}, \qquad K = \det \mathcal{W}$$

- Gauss-Bonnet

$$\iint_R K\,dA + \int_{\partial R} \kappa_g ds + \sum \text{turning angles} = 2\pi\chi(R)$$