The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2011-2012

Mathematics (Computational Methods) — 2 hours

*Answer **fou**r questions. If you answer more than four questions, only your best four will be counted.*
Graph paper is provided for question 5.

1.

(i) Briefly describe the classification of second order partial differential equations as elliptic, hyperbolic and parabolic and hence classify the equation $\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2}$ as one of these types. **(*5 marks*)**

(ii) Use Taylor's theorem to derive the forward and central difference formulae for the first derivative of a function $f(x)$ and also find a formula for the second derivative of $f(x)$. **(*6 marks*)**

(iii) Use the formulae of part (ii) to derive equations for an explicit approximate solution of the heat equation $\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2}$ on the interval where $0 \le x \le 1$ and $t \ge 0$ subject to the additional conditions $u(x,0) = 5x^2 + 11x,\ u(0,t) = 0,\ \frac{\partial u}{\partial x}(1,t) = 0.$ (The interval should be divided into 5 sections and a time step of 0.004 seconds should be used.) **(*7 marks*)**

(iv) Set up a table showing the values of $u(x,t)$ at the grid points for $t = 0$ and extend the table to include the values where $t \le 0.008$. **(*7 marks*)**

2.

(i) Let $A = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 4 \end{pmatrix}$.

Find the LU decomposition of A, where L is a lower triangular matrix with ones on the principal diagonal and U is an upper triangular matrix. **(*6 marks*)**

(ii) Given that $L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{7} & \frac{4}{7} & 1 \end{pmatrix}$, $U^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{7} & \frac{1}{24} \\ 0 & \frac{4}{7} & \frac{1}{6} \\ 0 & 0 & \frac{7}{24} \end{pmatrix}$, use these to find A^{-1}.

(*4 marks*)

(iii) The solution of the equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ is to be approximated in the cross-shaped region depicted in Figure 1

$$\begin{array}{ccccc} & 0 & 0 & 0 & \\ 2 & 4 & * & 4 & 2 \\ 4 & * & * & * & 4 \\ 2 & 4 & * & 4 & 2 \\ & 0 & 0 & 0 & \end{array}$$

Figure 1

where the numbers indicate the values of u on the boundary. Making use of any symmetry in the diagram suggest suitable notation for the unknown values of the variable and derive approximate formulae for the solution of the problem.

(*7 marks*)

(iv) Show that the equations are of the form $A\mathbf{u} = \mathbf{b}$ where A is the matrix in part (i) and $\mathbf{u}$ and $\mathbf{b}$ are column vectors. Hence, or otherwise, find the values of u at all the internal grid-points of the region.

(*8 marks*)

3.

(i) Briefly describe the strategy for fitting a cubic spline to a set of data of the form:

0	1	2	3	...	n
f_0	f_1	f_2	f_3	...	f_n

You should write down the initial equations, and indicate the order in which you would eliminate the variables, but you are not required to solve them in detail in this general case. Also indicate any additional information you would require.

(*8 marks*)

(ii) By suitably modifying the procedure, fit a cubic spline to the following data set:

x	0	1	3
f	15	30	40

together with the conditions that the **second** derivative at $x = 0$ is zero and that the **first** derivative at $x = 3$ is zero. Find the value of the resulting function when $x = 2$.

(*17 marks*)

4.

Let $f(x, y) = x^4 + 4y^2 - 4xy + 4x - 12y + 20$.

(i) Starting from the initial point $(x_0, y_0) = (0,0)$, use one iteration of the method of steepest descent to find a cubic equation whose solution will lead to the first iteration point (x_1, y_1). You are not required to solve this equation, but should verify that (x_1, y_1) is approximately $(-0.4137, 1.2411)$. **(*10 marks*)**

(ii) Find the gradient vector for the second iteration of this method, and verify that it is at right angles to the previous descent direction. **(*3 marks*)**

(iii) Use one iteration of Newton's method, starting from $(x_0, y_0) = (0,0)$ to search for the minimum point of $f(x, y)$. **(*6 marks*)**

(iv) By solving directly for the critical point find the actual minimum point of the function and check using a second derivative test that it is indeed a minimum value.

(*6 marks*)

5.

Demonstrate the branch and bound method for solving integer programming problems on the following example:

Maximise $z = 2x + 5y$ subject to the constraints:

$x \geq 0$, $y \geq 0$, $3x + 4y \leq 12$, $x + 3y \leq 6$, x, y are integers.

You may solve the associated linear programming problems by inspection of a graph, but the precise positions of the vertices should be calculated by solving appropriate simultaneous equations. You should also indicate the structure of your solution by drawing an appropriate tree diagram. (Simply evaluating the objective function at all feasible lattice points will gain very few marks)

(*25 marks*)

6.

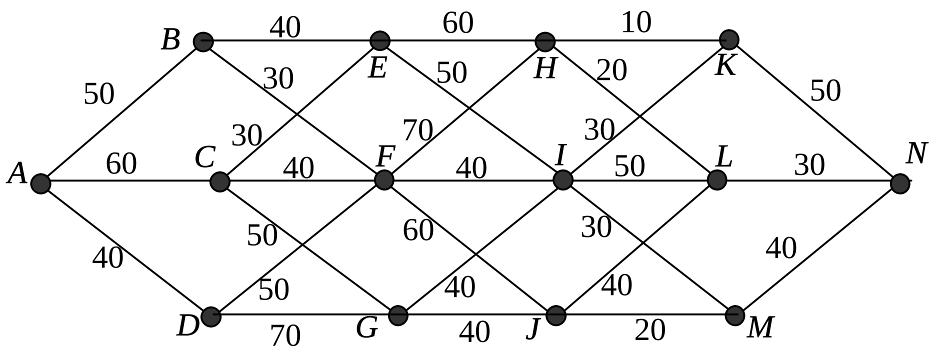

Figure 2

Figure 2 shows the driving time in minutes between key junctions of a road network. It is desired to find the quickest route from A to N using the roads shown.

(i) Explain how a dynamic programming approach can be used to solve this problem.

(*4 marks*)

(ii) Carry out the process to find the optimal route. **(*21 marks*)**

End of Question Paper