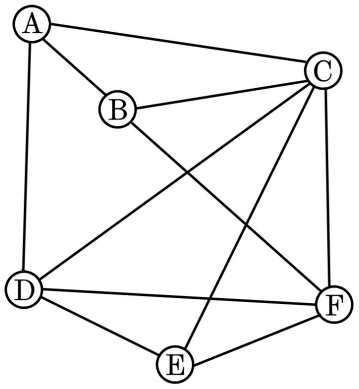
The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–12**

Graph Theory **2 hours 30 minutes**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Consider the following graph.



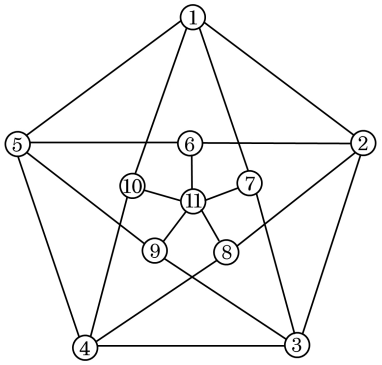
(a) Is this graph Eulerian? Explain your answer. ***(2 marks)***

(b) Is this graph planar? Explain your answer. ***(2 marks)***

(c) Is this graph Hamiltonian? Explain your answer. ***(2 marks)***

(d) Is this graph bipartite? Explain your answer. ***(2 marks)***

(ii) Consider the following graph.



(a) Is this graph planar? Explain your answer. ***(4 marks)***

(b) Show how this graph can be drawn on the Möbius band without any of its edges crossing. You should begin by drawing the cycle 1-7-3-9-5-6-2-8-4-10-1 as a regular polygon. Your drawing *must* use the given numbering of the vertices. ***(4 marks)***

(c) Show how this graph can be drawn on the torus without any of its edges crossing. You should begin by drawing the cycle 1-7-3-9-5-6-2-8-4-10-1 as a regular polygon. Your drawing *must* use the given numbering of the vertices. ***(4 marks)***

(iii) By drawing forests, determine the number of non-isomorphic forests with exactly 6 vertices and two trees. ***(5 marks)***

2 (i) (a) State Euler's formula for a plane, connected graph. ***(1 mark)***

(b) Let G be a simple connected planar graph with more than three vertices. Show that $e \leq 3v - 6$, where e is the number of edges of G and v is the number of vertices. ***(3 marks)***

(c) Now assume that G has no cycles of length three.
Show that $e \leq 2v - 4$. ***(2 marks)***

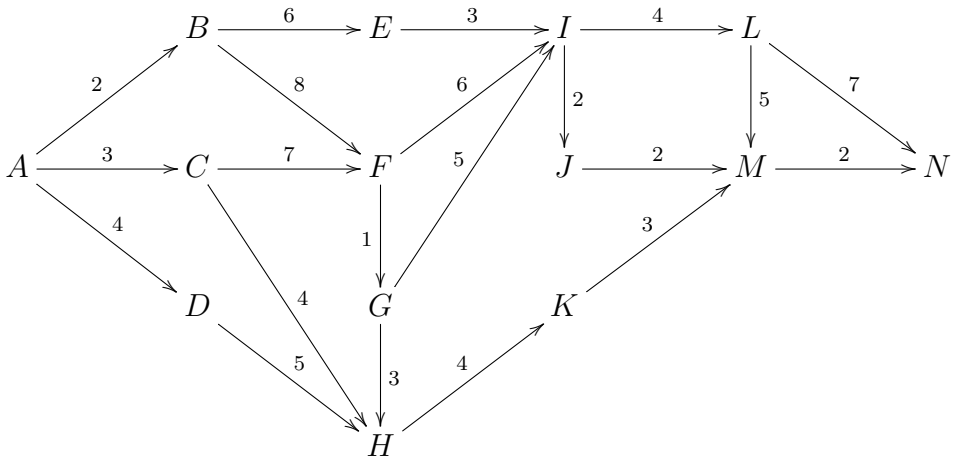
(d) Using the inequalities of parts (b) and (c), show that $K_{3,3}$ and K_5 are not planar. ***(2 marks)***

(ii) A plane connected graph has t faces of degree 3, s faces of degree 4, h faces of degree 6, no faces of any other degree and every vertex of degree 4. Express the number of vertices, edges and faces in terms of t, s and h and prove that $t = 2h + 8$. Draw such a graph with t, s and h all non-zero and give the values of t, s and h for your example. ***(10 marks)***

(iii) (a) State Ore's theorem on Hamiltonian graphs. ***(2 marks)***

(b) Show that a simple graph with 12 vertices and 57 edges must be Hamiltonian. ***(5 marks)***

3 (i) Consider the following diagram.



(a) Use the shortest and longest path algorithms to determine all shortest and longest paths from A to N. State the length s of the shortest paths and the length l of the longest paths. ***(14 marks)***

(b) Which length can neither be increased nor decreased, without altering l? ***(3 marks)***

(ii) The distances between six towns, U, V, W, X, Y, and Z are given in the table below.

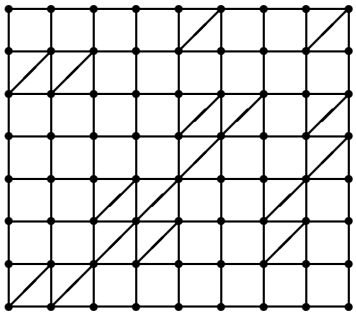
U					
4	V				
9	6	W			
15	11	12	X		
18	16	15	10	Y	
23	17	16	15	11	Z

(a) Starting at U, use "the heuristic algorithm for finding a good upper bound for the travelling salesman problem" on these towns. State the order in which you add the towns to the circuit. ***(4 marks)***

(b) By initially omitting U, give a good lower bound for the travelling salesman problem for these towns. ***(4 marks)***

4 (i) By drawing a suitable graph, determine whether the braced framework shown is rigid. If it is rigid, is it a minimum bracing? If it is not rigid, how many braces need to be inserted to make it rigid, and how many should *then* be removed to make it a minimum bracing? Justify all your answers.

(9 marks)



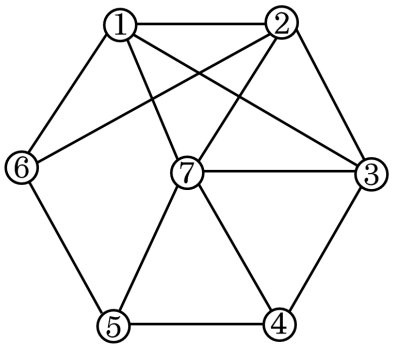
(ii) At a certain university in South Yorkshire, there are six groups of students, A-F. Each group has been assigned three employment workshops from a total of seven, labelled 1-7. Each workshop takes one whole day to complete. Your job is to timetable these workshops. By drawing a suitable graph, find the minimum number of days needed for each group to complete all their assigned workshops.

A :	1,3,5	D :	3,4,7
B :	2,4,5	E :	3,4,6
C :	1,3,6	F :	5,6,7

(9 marks)

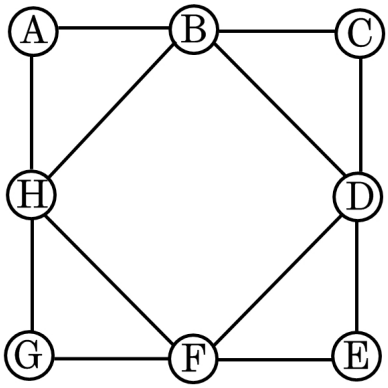
(iii) By using the planarity algorithm, decide whether the graph below is planar.

(7 marks)



5 (i) Let G be a graph with chromatic polynomial $k(k-1)(k-2)^2(k-4k+5)$. How many vertices and edges does this graph have? Define the chromatic number of a graph. What is the chromatic number of G? ***(5 marks)***

(ii) Consider the graph H shown below.



(a) What is the chromatic polynomial of H? ***(12 marks)***

(b) In how many ways can this graph H be coloured with four colours? ***(2 marks)***

(c) Define the chromatic index of a graph. What is the chromatic index of H? ***(4 marks)***

(d) What is the minimum number of colours needed to give a face colouring of the graph H? ***(2 marks)***

End of Question Paper