The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011–2012**

Knots and Surfaces **2 hours and 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

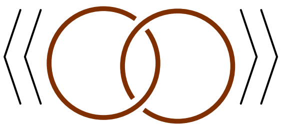
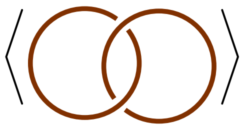
1 Define the terms *link universe* and *link diagram*. What does it mean when the word 'knot' is used instead of the word 'link'? ***(7 marks)***

Draw the *Reidemeister moves* and state *Reidemeister's Theorem*. ***(8 marks)***

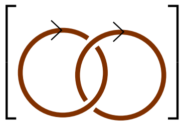
Describe a procedure by which any knot universe can be given a choice of crossings so that the resulting diagram is Reidemeister equivalent to the unknot. Illustrate this procedure on the standard trefoil universe and give a sequence of Reidemeister moves from that diagram to the standard unknot diagram. ***(7 marks)***

Prove that your procedure always gives the unknot. ***(3 marks)***

2 (i) Calculate the unnormalized bracket and the Kauffman bracket of the Hopf link:

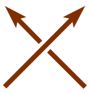
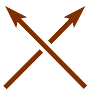
 and  .

Hence calculate the Jones polynomial of the negative Hopf link:

f  .

(15 marks)

(ii) In this question k_+, k_- and k_0 are link diagrams which are the same apart from in some disc where they differ as shown.

k_+ :  k_- :  k_0 : 

Show that if k_+ is a knot, then k_- is a knot and k_0 is a two-component link.

You are given that if l is any link diagram with b components then the Jones polynomial evaluated at 1 is given by $f[l]|_{A=1} = (-2)^{b-1}$. Show that if k_+ is a *knot* then

$$\left.\frac{\mathrm{d}f[k_+]}{\mathrm{d}A}\right|_{A=1} = \left.\frac{\mathrm{d}f[k_-]}{\mathrm{d}A}\right|_{A=1}.$$

Use Question 1 to deduce that for any *knot* k

$$\left.\frac{\mathrm{d}f[k]}{\mathrm{d}A}\right|_{A=1} = 0.$$

(10 marks)

[You may use the skein relation

$$A^4 f[k_+] - A^{-4} f[k_-] = (A^{-2} - A^2) f[k_0].]$$

3 (i) (a) What is a *compact connected surface*? What is *the connected sum* of two such surfaces? ***(6 marks)***

(b) List the *standard surfaces* together with their standard surface words. ***(5 marks)***

(c) Describe the classification of compact connected surfaces. ***(3 marks)***

(ii) (a) Draw pictures of the torus T and the Klein bottle K. Show that the surface words $aacbc^{-1}b^{-1}$ and $aabcbc^{-1}$ are equivalent using word operations. Hence or otherwise, show that the connected sum of a torus and the projective plane P is homeomorphic to the connected sum of the Klein bottle and the projective plane: $T\#P \cong K\#P$. ***(7 marks)***

(b) Find the connected sum of the orientable surface $M(g)$ and the non-orientable surface $N(h)$ where $g \geq 0, h \geq 1$. ***(4 marks)***

4 (i) (a) State the *inclusion/exclusion principle* for the Euler characteristic, and state the Euler characteristic of a *point*, a *line segment* and a *circle.* ***(6 marks)***

(b) Using these together with the homeomorphism invariance of the Euler characteristic, find the Euler characteristic of a *tube* and of a *torus.* ***(8 marks)***

(ii) (a) State the formula for the Euler characteristic $\chi(P)$ of a polyhedron P, explaining any symbols you use. ***(3 marks)***

(b) Suppose that P is a polyhedron made by gluing polygons together. For a vertex v, define the *deficit* β_v of v to be

$$\beta_v := 2\pi - \sum \alpha_i$$

where $\alpha_1, \ldots, \alpha_N$ are the angles of the corners meeting at the vertex v. The *total deficit* $\beta(P)$ of P is defined by

$$\beta(P) = \sum_v \beta_v$$

where the sum is over all the vertices of the polyhedron P.

Calculate the total deficit for a cube and for a regular tetrahedron.

Show that, in general,

$$\beta(P) = 2\pi\chi(P).$$

(8 marks)

End of Question Paper