The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011–2012**

MAS346 Groups and Symmetry **2 hours 30 minutes**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Let G and H be groups and let $f : G \to H$ be a function.
Explain what it means to say that f is an isomorphism, defining carefully any terminology you use. ***(4 marks)***

(ii) Explain why the inverse f^{-1} of a group isomorphism $f : G \to G$ is a group isomorphism, and the composition $f \circ g$ of two group isomorphisms $f, g : G \to G$ is a group isomorphism. Your explanation should include a detailed proof that f^{-1} and $f \circ g$ are group homomorphisms. ***(6 marks)***

(iii) (a) Define the automorphism group $\mathrm{Aut}(G)$ of a group G and prove that it is a group under composition of functions. ***(6 marks)***

(b) Let n be a positive integer. Which other group is $\mathrm{Aut}(\mathbf{Z}/n\mathbf{Z})$ isomorphic to? (No proof required.) ***(2 marks)***

(iv) Let a be any element of G. Prove that the map $\omega_a : G \to G$ defined by

$$x \mapsto \omega_a(x) := axa^{-1}$$

is an element of $\mathrm{Aut}(G)$. ***(7 marks)***

2 (i) Define the centre of a group and prove that it is a normal subgroup. ***(4 marks)***

(ii) Let $T_n(\mathbf{C})$ be the group of invertible $n \times n$ upper triangular matrices over $\mathbf{C}$, i.e. matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_{nn} \end{pmatrix} \in \mathrm{GL}_n(\mathbf{C}).$$

Prove that its centre is the set of scalar matrices $\{rE \,|\, r \in \mathbf{C}^*\}$, where E is the $n \times n$ identity matrix. ***(6 marks)***

(iii) Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group where 1 is the identity of Q and the multiplication is given by the rules:

$$i^2 = j^2 = k^2 = -1, \; (-1)a = a(-1) = -a \text{ for all } a \in Q,$$

$$(-1)^2 = 1, \; ij = -ji = k, \; jk = -kj = i, \; ki = -ik = j.$$

(a) Find all the subgroups of Q and their orders, and justify your response. ***(8 marks)***

(b) Find the centre $Z(Q)$ of the group Q. ***(2 marks)***

(c) Exhibit an isomorphism between the quotient group $Q/Z(Q)$ and $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. (You do not need to verify that your map is a homomorphism.) ***(5 marks)***

3 Let H and K be groups, $\phi : K \to Aut(H)$ be a group homomorphism, and $H \rtimes_\phi K$ be the semidirect product of H and K with respect to ϕ.

(i) Give the definition of the group $H \rtimes_\phi K$, i.e. describe the underlying set and the multiplication. ***(3 marks)***

(ii) Given an element $(a, x) \in H \rtimes_\phi K$, prove that $(a, x)^{-1} = (\phi(x^{-1})(a^{-1}), x^{-1})$. ***(5 marks)***

(iii) Now let H and K be subgroups of a group G such that $G = HK$, H is a normal subgroup of G, and $H \cap K = \{e\}$ where e is the identity of the group G.

Prove that $G \simeq H \rtimes_\psi K$ with $\psi : K \to Aut(H)$, $k \mapsto \psi(k)$, where $\psi(k)(h) := khk^{-1}$, $k \in K$, $h \in H$. ***(11 marks)***

(iv) Prove that S_4 is the semi-direct product of A_4 and the subgroup generated by the transposition (12). ***(6 marks)***

4 (i) Let X be a subset of $\mathbf{R}^n$. Define the symmetry group $\mathrm{Symm}(X)$ and the direct symmetry group $\mathrm{Dir}(X)$. ***(4 marks)***

(ii) Let G be a finite group acting on a set M. How is the order of the orbit of an element $m \in M$ under the action of G related to the order of G? ***(3 marks)***

(iii) Let C be a cube centred at the origin in $\mathbf{R}^3$ and write $H = \mathrm{Dir}(C)$.

(a) Describe a set of four elements on which H acts, and explain why this gives a homomorphism $\phi : H \to S_4$. ***(5 marks)***

(b) Give a geometric description of elements $g, h \in H$ such that $\phi(g) = (1\,2)$ and $\phi(h) = (1\,2\,3)$. ***(4 marks)***

(c) Let N be the set of faces of the cube C. Describe all the rotations stabilizing a face, and use (ii) to compute the order of H. ***(5 marks)***

(d) Describe the orbits of H on the set $N \times N$ of pairs of faces and determine their sizes. ***(4 marks)***

5 (i) State the Sylow theorems. You should carefully define all the terms and notation used. ***(5 marks)***

(ii) Let G be a group of order $5 \times 11 \times 13$.

(a) Show that G has normal subgroups of order 11 and 13 and 143. ***(7 marks)***

(b) By considering the conjugation action of G on the Sylow 13-subgroup show that the Sylow 13-subgroup lies in the centre of G. ***(7 marks)***

(iii) Give the definition of a simple group. ***(2 marks)***

(iv) Show that there is no simple group of order 12. ***(4 marks)***

End of Question Paper