The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011–2012**

Time Series **2 hours**

Marks will be awarded for your best ***three*** *answers.*

RESTRICTED OPEN BOOK EXAMINATION
Candidates may bring to the examination lecture notes and associated lecture material (but no textbooks) plus a calculator that conforms to University regulations.
There are 99 marks available on the paper.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) The figure below shows the quarterly averages of the Euro/£ and US$/£ exchange rates between 2001 and 2005 (source: Bank of England). Also plotted is the price of gold relative to its price in the first quarter of 2001. Describe the three time series and their relationship, using suitable technical terms and adding approximate quantification where appropriate. Detailed numerical comparisons of the series are not required. ***(9 marks)***

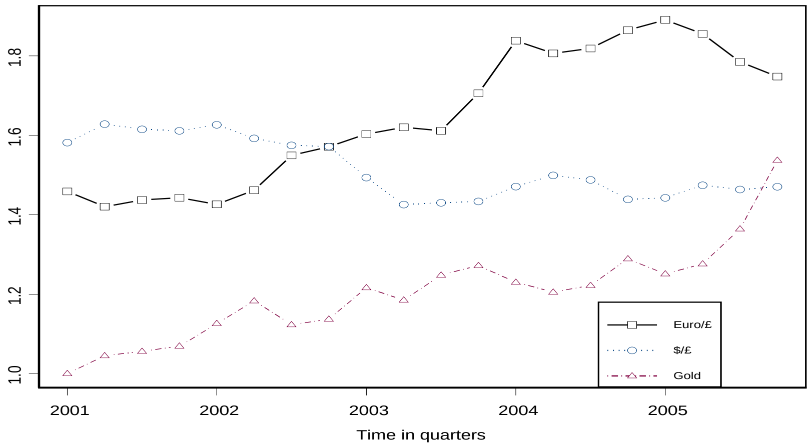

Figure 1: Dollar/Pound, Euro/Pound and Gold Index, Jan 2001 - Dec 2005

(ii) The following table shows the sample acf r_h and pacf $a_h^{(h)}$ values of the Dollar/pound series where the lag $h = 1, 2, 3, \ldots$ is in quarters of a year.

Lag h	1	2	3	4	5	6	7	8
r_h	0.9123	*	0.6287	0.4604	0.2703	0.0898	-0.0653	-0.2012
$a_h^{(h)}$	*	-0.3586	-0.0071	-0.2779	-0.1745	-0.0563	-0.0252	-0.0706

Calculate the two missing values. Without making any further calculations, describe briefly how using the above table, you could investigate whether the time series is stationary or not. ***(11 marks)***

(iii) Investigate and compare possible models for the series given the values in the table in (ii) and your calculations for the missing ones. What further evidence is desirable for a more complete identification of a suitable model? ***(13 marks)***

2 Consider the time series model

$$X_t = \frac{1}{3}X_{t-1} + \epsilon_t + \frac{1}{2}\epsilon_{t-1} - \frac{1}{4}\epsilon_{t-2},$$

where ϵ_t is white noise with variance 4.

(i) Explain why in this model X_t has zero mean. ***(4 marks)***

(ii) Show that this model is invertible. ***(5 marks)***

(iii) If $Var(X_t) = 9$, calculate the autocorrelation function (ACF) of X_t. ***(14 marks)***

(iv) Find a state space representation for the model for X_t. Write down the observation and evolution equations and state the distribution of the observation and evolution innovations. ***(10 marks)***

3 (i) Consider the AR(1) time series model

$$X_t = \alpha X_{t-1} + \epsilon_t,$$

where ϵ_t is a white noise sequence with variance σ^2.

(a) Derive the formula

$$X_{n+k} = \alpha^k X_n + \sum_{i=0}^{k-1} \alpha^i \epsilon_{n-i}$$

for some positive integers n and k. ***(6 marks)***

(b) Using (a) or otherwise, derive the k-step ahead forecast mean and variance of X_{n+k}, based on observed time series data $X_1 = x_1, \ldots, X_n = x_n$. ***(8 marks)***

(c) If ϵ_t is normally distributed, write down a 95% prediction interval for X_{n+k}. ***(3 marks)***

(ii) Consider the AR(2) time series model

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \epsilon_t,$$

where ϵ_t is as above a white noise sequence with variance σ^2.

(a) Write down X_t in state-space form. ***(2 marks)***

(b) Using the state-space form in (a), derive the 2-step ahead forecast mean and variance of X_{n+2}, based on observed time series data $X_1 = x_1, \ldots, X_n = x_n$. ***(10 marks)***

(c) Hence show that the 2-step forecast variance of X_{n+2} is the same as that of an AR(1) model defined by $X_t = \alpha_1 X_{t-1} + \epsilon_t$. ***(4 marks)***

4 A simple model for the size of the UK population of the lapwing (*Vanellus vanellus*, a wading bird of the plover family) is based on the evolution over years t of the number A_t of adult female birds and the number J_t of juvenile female birds. It is supposed that

$$A_t = \alpha(A_{t-1} + J_{t-1}) + \epsilon_t,$$

where α is a constant representing the year-to-year survival rate of birds and

$$J_t = \beta A_{t-1} + \eta_t,$$

where β is a fertility rate. Here (ϵ_t) and (η_t) are independent white noises having variances σ_ϵ^2 and σ_η^2, respectively. Each year a census of breeding pairs is carried out, but on average only a proportion $\gamma (0 < \gamma < 1)$ of pairs are spotted. The recorded census count X_t is assumed to be related to A_t by

$$X_t = \gamma A_t + \delta_t,$$

where (δ_t) is a white noise with variance σ_δ^2.

(i) If $\theta_t = (A_t, J_t)'$, the model can be described by the equations

$$\theta_{t+1} = G\theta_t + \omega_{t+1}, \tag{1}$$

$$X_t = F'\theta_t + \delta_t, \tag{2}$$

where G is a constant matrix, F is a constant vector and ω_t is a random vector. Find the values of F and G and write down the mean and covariance matrix of ω_t. ***(12 marks)***

(ii) Explain in terms of Lapwing population size, the ideas of *filtering* and X-*prediction*. If information is only known up to time $t-1$, which of these should be done first, in order to predict θ_t? ***(5 marks)***

(iii) Given information $x^{t-1} = (x_1, \ldots, x_{t-1})$, write down the equations giving the prior mean vector and covariance matrix of θ_t. Given information $x^t = (x_1, \ldots, x_t)$, write down the posterior mean vector and covariance matrix of θ_t, for the Lapwing model. ***(5 marks)***

(iv) You are given that $\alpha = 0.3, \beta = 0.5, \gamma = 0.7$ and $\sigma_\epsilon^2 = \sigma_\eta^2 = \sigma_\delta^2 = 40$. You also know that the Lapwing population at time zero is normally distributed with mean $(90, 0)^T$ and covariance $\begin{pmatrix} 75 & 0 \\ 0 & 0 \end{pmatrix}$.

(a) Give the distribution of the 1-step ahead prediction error at time $t = 1$. ***(7 marks)***

(b) If the observed number of Lapwings at time $t = 1$ is 26 calculate the actual 1-step ahead prediction error. ***(4 marks)***

End of Question Paper