The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–2012**

Topics in Advanced Fluid Mechanics **2 hours 30 minutes**

Marks will be awarded for your best ***four*** *answers.*

1 Consider the 3D Navier-Stokes equations (written with standard notations) for an incompressible fluid of a constant unit density in the whole space $\mathbb{R}^3$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u},$$

$$\nabla \cdot \boldsymbol{u} = 0.$$

We assume the fluid is at rest at infinity and spatial averages of $\boldsymbol{u}$ and p are zero.

(1) Derive a set of equations for the impulse defined by $\boldsymbol{\gamma} = \boldsymbol{u} + \nabla\phi$,

$$\frac{D\boldsymbol{\gamma}}{Dt} = -(\nabla \boldsymbol{u})^T \boldsymbol{\gamma} + \nabla\lambda + \nu\nabla^2\boldsymbol{\gamma},$$

$$\frac{D\phi}{Dt} = p - \frac{|\boldsymbol{u}|^2}{2} + \lambda + \nu\nabla^2\phi,$$

where λ is an arbitrary scalar function of $\boldsymbol{x}$ and t. ***(13 marks)***

(2) In the inviscid case $\nu = 0$, by using a special choice of $\lambda = 0$ (geometric gauge) show that the function

$$f(t) = \int_{\mathbb{R}^3} \phi d\boldsymbol{x}$$

satisfies

$$\frac{df}{dt} = -\int_{\mathbb{R}^3} \frac{|\boldsymbol{u}|^2}{2} d\boldsymbol{x}$$

and solve this equation for $f(t)$ in terms of $f(0)$.

(12 marks)

2 (1) Consider an infinitesimal material line element $\boldsymbol{l}$ subject to an incompressible velocity field $\boldsymbol{u}$. Show that it satisfies a dynamical equation of the form

$$\frac{D\boldsymbol{l}}{Dt} = (\boldsymbol{l} \cdot \nabla)\boldsymbol{u}.$$

(5 marks)

(2) Consider two infinitesimal material line elements $\boldsymbol{l}, \boldsymbol{m}$ subject to $\boldsymbol{u}$ to form a surface element by $\boldsymbol{A} = \boldsymbol{l} \times \boldsymbol{m}$. Show that $\boldsymbol{A}$ satisfies the following equation

$$\frac{D\boldsymbol{A}}{Dt} = -(\nabla \boldsymbol{u})^T \cdot \boldsymbol{A}.$$

(10 marks)

(3) Show that

$$\frac{D}{Dt}(\boldsymbol{l} \cdot \boldsymbol{A}) = 0$$

and give an interpretation to this conservation. ***(5 marks)***

(4) Consider a more material surface, that is, a surface defined by

$$f(x, y, z, t) = \text{constant}$$

which moves with the fluid:

$$\frac{\partial f}{\partial t} + (\boldsymbol{u} \cdot \nabla)f = 0.$$

Derive an equation for the normal vector to the surface and confirm that it has the same form as the above equation for $\boldsymbol{A}$. ***(5 marks)***

3 Consider a model equation for vorticity ω defined in $\mathbb{R}^1$:

$$\frac{\partial \omega}{\partial t} = \omega H[\omega] - \gamma\omega,$$

with an initial condition

$$\omega(x, t = 0) = \omega_0(x).$$

Here $\gamma(> 0)$ denotes damping and $H[\omega]$ the Hilbert transform

$$H[\omega](x) = \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{\omega(y)}{x - y} dy,$$

where $⨍$ is a principal value integral.

(1) Derive an equation for $H[\omega](x)$ as

$$\frac{\partial H[\omega]}{\partial t} = \frac{1}{2}\left(H[\omega]^2 - \omega^2\right) - \gamma H[\omega].$$

(Hint: $H[fg] = H[f]g + fH[g] + H[H[f]H[g]]$, $H[H[f]] = -f$ for any f, g.)

(4 marks)

(2) Derive an equation for $f(x) = \omega(x) + iH[\omega](x)$ as

$$\frac{\partial f(x, t)}{\partial t} = -\frac{i}{2} f(x, t)^2 - \gamma f(x, t).$$

(4 marks)

(3) By introducing a new function $g(x, t) = e^{\gamma t} f(x, t)$, derive an equation for it as

$$\frac{\partial g(x, t)}{\partial t} = -\frac{i}{2} e^{-\gamma t} g(x, t)^2.$$

(4 marks)

(4) By defining a new time $s = (1 - e^{-\gamma t})/\gamma$ and $\tilde{g}(x, s) = g(x, t)$, show that

$$\frac{\partial \tilde{g}(x, s)}{\partial s} = -\frac{i}{2} \tilde{g}(x, s)^2.$$

By solving it derive an explicit solution for $\omega(x, t)$

$$\omega(x, t) = e^{-\gamma t} \frac{\omega_0(x)}{(1 - \frac{s}{2} H[\omega_0])^2 + (\frac{s}{2}\omega_0(x))^2},$$

with s defined as above. ***(10 marks)***

(5) Show that the solution breaks down at a zero of $\omega_0(x)$ at a time

$$t_* = \frac{1}{\gamma} \log\left(1 - \frac{2\gamma}{\max_x H[\omega_0]}\right).$$

(3 marks)

4 We consider a system of N point-vortices:

$$\frac{dz_j}{dt} = -\frac{1}{2\pi i} {\sum_{m=1}^{N}}' \frac{\kappa_m}{z_j^* - z_m^*}, \quad (j = 1, 2, \ldots, N)$$

where $\sum{}'$ denotes a summation excluding $j = m$, $*$ complex conjugate, $z_j = x_j + iy_j$ and (x_i, y_i), $i = 1, 2, \ldots, N$ coordinates of a point vortex of strength κ_i.

(1) Using the complex notation, show that $\sum_{j=1}^{N} \kappa_j z_j$ and $\sum_{j=1}^{N} \kappa_j z_j z_j^*$ are constants of motion, that is, they are independent of time.

(10 marks)

(2) Show that the above set of equations can be rewritten in the following form

$$\kappa_j \frac{dz_j^*}{dt} = 2i \frac{\partial H}{\partial z_j},$$

where

$$H = \frac{1}{4\pi} {\sum_{j,k=1}^{N}}' \kappa_j \kappa_k \log \frac{1}{|z_j - z_k|}$$

is the Hamiltonian function. ***(10 marks)***

(3) Show that H is also a constant of motion by regarding H as a function of z_j and z_j^* $(j = 1 \ldots N)$. ***(5 marks)***

5 Consider a velocity field of the form

$$\boldsymbol{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, w \right),$$

where ψ and w are functions of x, y only.

(1) Show that the vorticity field takes the following form

$$\boldsymbol{\omega} = \left(\frac{\partial w}{\partial y}, -\frac{\partial w}{\partial x}, -\nabla^2 \psi \right).$$

(4 marks)

(2) Show that the helicity density takes the following form

$$\boldsymbol{u} \cdot \boldsymbol{\omega} = \nabla \psi \cdot \nabla w - w \nabla^2 \psi.$$

(4 marks)

(3) Show that

$$\boldsymbol{u} \times \boldsymbol{\omega} = \left(\frac{\partial \psi}{\partial x} \nabla^2 \psi + \frac{\partial}{\partial x} \left(\frac{w^2}{2} \right), \frac{\partial \psi}{\partial y} \nabla^2 \psi + \frac{\partial}{\partial y} \left(\frac{w^2}{2} \right), \frac{\partial(\psi, w)}{\partial(x, y)} \right),$$

where $\frac{\partial(\psi, w)}{\partial(x, y)}$ denotes the Jacobian determinant between ψ and w. ***(4 marks)***

(4) Consider the 3D Euler equations for an incompressible fluid

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{u} \times \boldsymbol{\omega} - \nabla \left(p + \frac{|\boldsymbol{u}|^2}{2} \right).$$

You are given that, under the conditions of steady flows ($\partial / \partial t = 0$), $w = C(\psi)$, $p + \frac{|\boldsymbol{u}|^2}{2} = H(\psi)$, where C and H are arbitrary functions of ψ. Derive the following formula which expresses the vorticity in terms of $C(\psi)$, $H(\psi)$:

$$\nabla^2 \psi = \frac{dH}{d\psi} - C \frac{dC}{d\psi}.$$

(13 marks)

End of Question Paper