The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011–2012**

MAS422 Magnetohydrodynamics **2 hours**

Answer all four questions.

1 (i) Sketch the field lines for the following fields. Indicate clearly the direction of the field in each case.

(a) $\mathbf{B} = (-x,\, y,\, 0)$ ***(5 marks)***

(b) $\mathbf{B} = B_0(y, \cos x, 0)$ ***(6 marks)***

(ii) Using solenoidal condition show that the following magnetic field has no sources or sinks of flux and determine its current

$$\mathbf{B}(r, \phi, z) = B_0\left(r, r^4\mathrm{e}^{-r^2}, -2z\right)$$

(6 marks)

(iii) Consider the magnetic induction equation in the case where the magnetic diffusivity $\eta = 0$. Use the standard vector identity

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$

together with Maxwell's equation $\nabla \cdot \mathbf{B} = 0$ and the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

to show that the induction equation may be written as

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{B}}{\rho}\right) + (\mathbf{u} \cdot \nabla)\frac{\mathbf{B}}{\rho} = \left(\frac{\mathbf{B}}{\rho} \cdot \nabla\right)\mathbf{u}\ .$$

(8 marks)

2 (i) Consider a horizontal magnetic field $B(z)\hat{\mathbf{x}}$ in equilibrium with a plasma, satisfying

$$O = -\frac{d}{dz}\left(p + \frac{B^2}{2\mu}\right) - \rho g$$

where $\rho = p/RT$.
If $T = T_0$ and $B(z) = B_0\frac{z}{H}$, where T_0, B_0, $H(= RT/g)$ are constants, show that

$$\frac{dp}{dz} + \frac{p}{H} = -cz$$

where $c = \frac{B_0^2}{\mu H^2}$.
Solve this equation for $p(z)$ if $p(0) = p_0$ in terms of H, p_0, $\beta = \frac{2\mu p_0}{B_0^2}$.

(11 marks)

(ii) Consider the effect of a flow

$$\mathbf{v} = -x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$$

on a magnetic field $B(x,t)\hat{\mathbf{y}}$. If the field is initially $(1 - x^2)\hat{\mathbf{y}}$, find $B(x,t)$ by using the method of characteristics to solve the induction equation in the perfectly conducting limit. For $t = 0$ and $t = 1$, sketch the function $B(x,t)$ and the field lines that lie initially between $x = -1$ and $x = 1$.

(11 marks)

(iii) Consider magnetic field in cylindrical polar coordinates (r, ϕ, z). If a magnetic field $\mathbf{B} = \mathbf{B}(r)$ varies with r alone, why can it not possess a radial component (B_r)? ***(3 marks)***

3 (i) If a plasma is incompressible and the radius of a flux tube is decreased by a factor 2, use conservation of mass and flux to determine what happens to its length and field strength? ***(8 marks)***

(ii) Given a velocity field $\mathbf{u} = (yz, -xz, 0)$ and the initial magnetic field $\mathbf{B}(\mathbf{x}, 0) = (x, -y, 0)$, find $\mathbf{B}(\mathbf{x}, t)$ by obtaining the Lagrangian coordinates corresponding to $\mathbf{u}$ and applying the Cauchy solution.

(17 marks)

4 (i) An inviscid, perfectly conducting, incompressible fluid, is permeated by a uniform magnetic field $\mathbf{B_0}$. The motion of the fluid is described by the momentum equation

$$\rho\left[\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{U}\right] = -\nabla p \ + \ \frac{(\nabla\times\mathbf{B})}{\mu_0}\times\mathbf{B}$$

and magnetic induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{U}\times\mathbf{B}).$$

The fluid is initially at rest (i.e. $\mathbf{U}_0 = 0$) and then given a small perturbation. Write down the linearised momentum and induction equations.

(7 marks)

(ii) Seeking solutions proportional to $\exp i(\mathbf{k}\cdot\mathbf{x} - \omega t)$ in the above linearised equations, show that

$$\omega^2 = \frac{(\mathbf{k}\cdot\mathbf{B_0})^2}{\mu_0\rho}$$

where μ_0 is the magnetic permeability and ρ the fluid density.

(10 marks)

(iii) Starting from Maxwell's equations in the MHD approximation and the generalised Ohm's law, derive the induction equation for the case where the conductivity σ is not a constant, to show that

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \eta\nabla^2\mathbf{B} - \nabla\eta\times(\nabla\times\mathbf{B}).$$

(8 marks)

End of Question Paper

Formulae Sheet

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

	u	v	w	f	g	h
cartesian	x	y	z	1	1	1
spherical	r	θ	ϕ	1	r	$r\sin\theta$
cylindrical	r	ϕ	z	1	r	1

$$\nabla \cdot \mathbf{V} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghV_u) + \frac{\partial}{\partial v}(fhV_v) + \frac{\partial}{\partial w}(fgV_w)\right]$$

$$\nabla \times \mathbf{V} = \frac{1}{gh}\left[\frac{\partial}{\partial v}(hV_w) - \frac{\partial}{\partial w}(gV_v)\right]\hat{u} \; + \; \frac{1}{fh}\left[\frac{\partial}{\partial w}(fV_u) - \frac{\partial}{\partial u}(hV_w)\right]\hat{v} + \frac{1}{fg}\left[\frac{\partial}{\partial u}(gV_v) - \frac{\partial}{\partial v}(fV_u)\right]\hat{w}$$