The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–12**

Analytic Number Theory **2 hours 30 minutes**

*Answer **Question 1** and three other questions. You are advised **not** to answer more than three of the questions 2 to 5: if you do, only your best three will be counted.*

1 (i) Given a character χ of the group $(\mathbb{Z}/N\mathbb{Z})^\times$, explain how one defines the associated *Dirichlet L-function* $L(s,\chi)$, and indicate its region of convergence. Describe, without proof, the behaviour of $L(1,\chi)$ and $\lim_{\sigma\to 1+} \sum \frac{\chi(p)}{p^\sigma}$.
(7 marks)

(ii) This question asks you to illustrate the proof of Dirichlet's Theorem in a specific case.

(a) List the characters of $(\mathbb{Z}/12\mathbb{Z})^\times$, indicating which are the non-trivial characters. ***(5 marks)***

(b) For each non-trivial character χ on your list, prove that $0 < L(1,\chi) < 2$. ***(6 marks)***

(c) Prove that there are infinitely many primes congruent to $7 \pmod{12}$.
(7 marks)

2 (i) State the Prime Number Theorem. ***(1 mark)***

We define the *prime-squared counting function* S as follows: For $x > 0$, we let $S(x)$ to be the number of primes p such that $p^2 \leq x$.

(a) Show that $S(x) \sim \dfrac{2\sqrt{x}}{\log x}$. ***(4 marks)***

(b) Let $a > b > 0$ be fixed positive real numbers. Calculate

$$\lim_{x\to\infty} \frac{S(ax)}{S(bx)},$$

and deduce that we can find two primes p and q such that $b < \dfrac{p^2}{q^2} < a$. ***(9 marks)***

Furthermore, show that there are infinitely many prime numbers p such that p^2 begins with the digits $2012\ldots$. ***(5 marks)***

(ii) Let $p_1, \ldots, p_n$ be n distinct primes. For any positive integer x, let $N_n(x)$ be the number of integers between 1 and x whose prime divisors belong to the set $\{p_1, \ldots, p_n\}$. Show that $N_n(x) \leq 2^n\sqrt{x}$. ***(6 marks)***

3 (i) Show that $\lfloor x+y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor = 0$ or 1 for all $x, y \in \mathbb{R}$. (Recall that $\lfloor x \rfloor$ is the greatest integer not exceeding x.) ***(3 marks)***

Now let n be a positive integer, and let p be a prime.

(a) Write down a formula for the highest power p that divides $n!$. Hence find all positive integers n for which the decimal expansion of $n!$ ends in exactly 25 zeroes. ***(7 marks)***

(b) Show that if p^r divides $\dbinom{2n}{n}$ then $p^r \leq 2n$. ***(4 marks)***

(ii) State Bertrand's Postulate. ***(1 mark)***

Let n be a positive integer.

(a) Use Bertrand's Postulate to show that $2n+m$ is a prime number for some odd number m with $0 < m < 2n$. ***(3 marks)***

(b) Let $0 \leq k < n$ and suppose $2n + (2k+1)$ is a prime number, say p. Show that the set $\{2k+1, 2k+2, \ldots, 2n-1, 2n\}$ can be partitioned into $n-k$ pairs (i, j) with $i + j = p$. ***(2 marks)***

(c) Use strong induction and the parts above to prove that the set $\{1, 2, \ldots, 2n\}$ can be partitioned into n pairs with each pair adding up to a prime number. ***(5 marks)***

4 (i) Let $f, g, h : \mathbb{N} \longrightarrow \mathbb{C}$ be three arithmetic functions related by the following formal product of Dirichlet series:

$$\left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{g(n)}{n^s}\right) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}.$$

(a) Write down the relation between f, g and h. The arithmetic function h is often called the *Dirichlet product* of f and g. ***(2 marks)***

(b) State a necessary and sufficient condition (on f) for the Dirichlet series $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ to have a (formal) Euler product expansion, and write down the Euler product expansion when that condition is met. Deduce that if $f, g : \mathbb{N} \longrightarrow \mathbb{C}$ are multiplicative functions then their Dirichlet product h is also multiplicative. ***(5 marks)***

(ii) Now let $d(n)$ be the number of (positive) divisors of n. You may assume that $d(n), d(n)^3$ are multiplicative functions.

(a) Write down the Euler product expansion of the Riemann zeta function $\zeta(s)$. Show that $\sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta(s)^2$. ***(4 marks)***

(b) By considering the divisors of p^k, derive a formula for $d(p^k)$ if p is a prime number and $k \geq 1$. ***(2 marks)***

(c) Give an indication of how you would derive the series expansion

$$\frac{1 + 4x + x^2}{(1 - x)^4} = 1 + 2^3 x + 3^3 x^2 + 4^3 x^3 + \ldots$$

given the identity

$$\frac{1 + x}{(1 - x)^3} = 1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \ldots = \sum_{n=0}^{\infty} (n + 1)^2 x^n.$$

You do not need to carry out any calculation. ***(2 marks)***

(d) Deduce, using the preceding parts, that

$$\sum_{n=1}^{\infty} \frac{d(n)^3}{n^s} = \zeta(s)^4 \prod_{\text{primes } p} \left(1 + \frac{4}{p^s} + \frac{1}{p^{2s}}\right).$$

(4 marks)

(e) Prove that

$$\sum_{m|n} d(m)^3 = \Big(\sum_{m|n} d(m)\Big)^2$$

for any positive integer n. (**Hint:** You might like to reduce the problem to the case when n is a power of a prime. If required you can use the identity $1^3 + 2^3 + \ldots + N^3 = (1 + 2 + \ldots + N)^2$ without proof.) ***(6 marks)***

5 (i) Recall that the Bernoulli polynomials $B_k(x)$ for $k \geq 0$ are defined by the generating series

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.$$

(a) Show that $\frac{d}{dx} B_k(x) = k B_{k-1}(x)$ and $\int_0^1 B_k(x)\, dx = 0$ for all $k \geq 1$. Hence, starting off with $B_0(x) = 1$, calculate $B_1(x)$, $B_2(x)$ and $B_3(x)$.

(10 marks)

(b) Derive the Fourier series expansion of $B_3(x)$ in the interval $[0, 1]$ given that

$$B_2(x) = 4 \sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{(2\pi n)^2}, \quad 0 \leq x \leq 1.$$

Hence evaluate the series

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \cdots$$

(6 marks)

(ii) State and prove the Euler-Maclaurin summation formula. ***(9 marks)***

End of Question Paper