The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011–2012**

Algebraic Topology **2 hours 30 minutes**

Answer ***all five*** *questions. All questions carry equal weight. You will be marked on correctness/completeness and rigour/presentation. Justify all your answers.*

1 Construct a topological space with fundamental group $\mathbb{Z}_2 \times \mathbb{Z}_2$, and classify its connected covering spaces.

2 Compute the fundamental group and homology groups of the 2-complex X obtained from D^2 by first deleting the interiors of two disjoint subdisks and then identifying all three boundary circles via homeomorphisms preserving clockwise orientations of these circles. Comment on the relationship between your answers. Use your answers to determine whether or not the resulting space is a surface.

3 (i) Why do we need the notion of reduced homology?

(ii) Express reduced homology in terms of relative homology.

(iii) Compute the relative homology of the following pairs (X, A). Use a fundamental group to check your answers at dimension 1:

(a) $X =$ unit interval, $A =$ endpoints

(b) $X = S^2$, $A =$ a pair of antipodal points

4 Let X be a finite-dimensional CW-complex with n-skeleton X^n for each n. Consider the long exact sequence of singular homology sequence for the pair (X^n, X^{n-1}).

(i) Show that $H_k(X^n) = 0$ for all $k > n$ (here H_k is singular homology).

(ii) Show that the inclusion $i : X^n \hookrightarrow X$ induces an isomorphism

$$i_* : H_k(X^n) \longrightarrow H_k(X)$$

if $k < n$.

(iii) Show that these facts are "obvious" for simplicial homology.

5 (i) What does it mean for two maps of spaces to be homotopic? Show that this is an equivalence relation.

(ii) What does it mean for two chain maps of chain complexes to be chain homotopic? Show that this is also an equivalence relation.

(iii) Discuss the importance of homotopy invariants, with examples.

End of Question Paper