The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2011–2012

MAS436: Functional Analysis — **2 hours 30 minutes**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

Throughout this paper, unless otherwise stated, all normed vector spaces and Hilbert spaces are either over the field of real numbers, $\mathbb{R}$, or the field of complex numbers, $\mathbb{C}$

1 (i) Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$. Let $C(\mathbb{D})$ be the vector space of continuous functions $f\colon \mathbb{D} \to \mathbb{C}$. Prove that we have a norm on $C(\mathbb{D})$ defined by the formula

$$\|f\| = \sup\{|f(z)| \mid z \in \mathbb{D}\}$$

and that $C(\mathbb{D})$ is complete with respect to this norm. ***(14 marks)***

(ii) State the Stone-Weierstrass theorem. ***(3 marks)***

(iii) Which of the following are dense subsets of $C(\mathbb{D})$? Justify your answer fully.

(a) The set of all complex polynomial functions $p\colon \mathbb{D} \to \mathbb{C}$.

(b) The set of all complex polynomial functions $p\colon \mathbb{D} \to \mathbb{C}$ of even degree.

(c) The set of all continuous functions $f\colon \mathbb{D} \to \mathbb{C}$ that extend to holomorphic functions in an open set containing $\mathbb{D}$.

(8 marks)

2 (i) (a) Let V be a complex normed vector spaces. Say what is meant when we describe a linear map $f: V \to \mathbb{C}$ as *bounded*. ***(2 marks)***

(b) Prove that a linear map $f: V \to \mathbb{C}$ is bounded if and only if it is continuous. ***(7 marks)***

(c) State the Hahn-Banach theorem. ***(3 marks)***

(ii) (a) Say what is meant when we say a map $g: \mathbb{C} \to V$ is holomorphic. ***(2 marks)***

(b) Let $f: V \to \mathbb{C}$ be a bounded linear map, and let $g: \mathbb{C} \to V$ be holomorphic. Show that $f \circ g$ is holomorphic, If, further, $g'(z) = 0$, show that $(f \circ g)'(z) = 0$. ***(5 marks)***

(c) Prove that if $g: \mathbb{C} \to V$ is holomorphic, with $g'(z) = 0$ for all $z \in \mathbb{C}$, then g is constant. ***(6 marks)***

If you wish, you may use without proof the fact that if $h: \mathbb{C} \to \mathbb{C}$ is holomorphic with $h'(z) = 0$ for all $z \in \mathbb{C}$, then h is constant.

3 (i) Say what is meant by an orthonormal sequence in a Hilbert space, and what is meant by saying that an orthonormal sequence is an orthonormal basis. ***(4 marks)***

(ii) Let H be a Hilbert space, and let (v_n) be an orthonormal sequence in H. Let $v \in H$. Suppose that

$$\|v\|^2 = \sum_{n=0}^{\infty} |\langle e_n, v\rangle|^2.$$

Prove that $v \in \overline{Span(v_n)}$. ***(8 marks)***

[Hint: Look at the norm of $w_N = v - \sum_{n=1}^{N} \langle e_n, v\rangle e_n$.]

(iii) Let

$$v_0 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{4}}, -\frac{1}{\sqrt{8}}, -\frac{1}{\sqrt{16}}, \dots\right).$$

Define a bounded lineear operator $S: l^2 \to l^2$ be the formula $S(a_1, a_2, a_3, \dots) = (0, a_1, a_2, a_3, \dots)$. Let $v_n = S^n v_0$.

Let (e_n) be the standard basis for the space l^2. Compute the sum

$$\sum_{n=0}^{\infty} |\langle e_n, v_k\rangle|^2.$$

(4 marks)

(iv) Show that the sequence (v_n) is an orthonormal basis for l^2. ***(9 marks)***

4 (i) Let H be a Hilbert space. Let $w \in H$. Show that

$$\|w\| = \sup\{|\langle v, w\rangle| \mid v \in H, \ \|v\| \leq 1\}.$$

(4 marks)

You may use the Cauchy-Schwarz inequality without proof.

(ii) State the *Riesz representation theorem* for bounded linear functionals on a Hilbert space. ***(2 marks)***

(iii) Let $T\colon H \to H$ be a bounded linear map. Prove that there is a unique bounded linear map $T^*\colon H \to H$ such that $\langle T^* v, w\rangle = \langle v, Tw\rangle$ for all $v, w \in H$.

Show further that $\|T^*\| = \|T\|$. ***(8 marks)***

(iv) Let $T\colon H \to H$ be a bounded linear map. Show that

$$T[H]^\perp = \ker T^*.$$

(3 marks)

(v) Define bounded linear maps $A, B\colon l^2 \to l^2$ by the formulae

$$A(a_1, a_2, a_3, a_4 \ldots) = (0, a_1, \frac{a_2}{2}, \frac{a_3}{3}, \frac{a_4}{4}, \ldots)$$

and

$$B(a_1, a_2, a_3, a_4 \ldots) = (a_2, \frac{a_1}{2}, a_4, \frac{a_3}{2}, a_6, \frac{a_5}{2}, \ldots)$$

respectively. Compute A^* and B^*. ***(8 marks)***

5 (i) Let A be a unital complex Banach algebra, and let $x \in A$. Prove that if $\|x\| < 1$, then the element $1 - x$ is invertible. ***(8 marks)***

(ii) Say what is meant by the spectrum, $Spectrum(x)$ of an element $x \in A$. ***(2 marks)***

(iii) Let $x \in A$. Prove that $Spectrum(x)$ is compact. ***(8 marks)***

(iv) Let $x \in A$ be an *involution*, that is to say an element where $x^2 = x$. Show that $Spectrum(x) \subseteq \{0, 1\}$. ***(7 marks)***

6 (i) Let $f, g \in L^1(\mathbb{R})$ be continuous functions. Show that we have a well-defined function $f * g \in L^1(\mathbb{R})$ defined by the formula

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t)g(t)dt.$$

(6 marks)

(ii) For continuous functions $f, g \in L^1(\mathbb{R})$, show that, when taking the Fourier transform, $\widehat{f * g}(\omega) = \hat{f}(\omega)\hat{g}(\omega)$. ***(6 marks)***

(iii) Let $\alpha > 0$. Let $f \in L^1(\mathbb{R})$ be continuous. Let $g(x) = f(\alpha x)$. Show that, when considering the Fourier transform, we have $\hat{g}(\omega) = \frac{1}{\alpha}\hat{f}(\frac{\omega}{\alpha})$. ***(3 marks)***

(iv) Let $\sigma > 0$. Let

$$g_\sigma = \frac{1}{\sigma} \exp\left(\frac{-x^2}{2\sigma^2}\right).$$

(a) Calculate the Fourier transform $\hat{g}_\sigma$. ***(6 marks)***

(b) Show that for any $\sigma, \tau > 0$, we have $g_\sigma * g_\tau = g_{\sqrt{\sigma^2+\tau^2}}$. ***(4 marks)***

You may assume here, if desired, the Fourier inversion theorem.

End of Question Paper