The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011–2012**

Optics and Symplectic Geometry **2 hours 30 minutes**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

Throughout the paper I *denotes an identity matrix and* J *denotes a matrix of the form* $\left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$. *All matrices have real entries. The standard symplectic form* Ω *on* $\mathbb{R}^{2n}$ *is defined by* $\Omega(Z, Z') = Q \cdot P' - P \cdot Q'$, *where* $Z = (Q, P)$ *and* $Z' = (Q', P')$ *are elements of* $\mathbb{R}^{2n}$.

In Questions 2 *to* 5 *you may, if you wish, use results from Question* 1.

1 (i) (a) Define what it means for a $2n \times 2n$ matrix S to be symplectic. **(2 marks)**

(b) Prove that the $2n \times 2n$ matrix J is invertible. **(2 marks)**

Let $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a $2n \times 2n$ matrix in block form, where A, B, C and D denote $n \times n$ matrices.

(c) Prove that S is symplectic if and only if the three equations

$$A^{\mathsf{T}}C = C^{\mathsf{T}}A, \qquad B^{\mathsf{T}}D = D^{\mathsf{T}}B, \qquad A^{\mathsf{T}}D - C^{\mathsf{T}}B = I,$$

hold. **(4 marks)**

(d) Assume that S is symplectic. Show that it is invertible and establish a formula for S^{-1} in block form. **(3 marks)**

(ii) (a) List the three properties which a map $\omega\colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ must have in order to be a symplectic form on $\mathbb{R}^{2n}$. **(3 marks)**

(b) Define the notion of symplectic basis for $(\mathbb{R}^{2n}, \omega)$. where ω is any symplectic form on $\mathbb{R}^{2n}$. **(3 marks)**

(c) Let $E_1, \dots, E_n, F_1, \dots F_n$ be vectors in $\mathbb{R}^{2n}$. Show that they form a symplectic basis of $(\mathbb{R}^{2n}, \Omega)$ if and only if the matrix with columns $E_1, \dots, E_n, F_1, \dots, F_n$ is symplectic. **(8 marks)**

2 (i) Consider a beach, with a straight coastline, as shown in Figure 1. Person M, standing on the beach, sees person S in the water, struggling and getting into difficulties. M can run at speed v_1 and swim at speed v_2. M, being a mathematician, runs towards a point P so as to reach S in the shortest possible time. Prove that the angles which M's path on land and in the water will make with the normal to the beach are related by

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

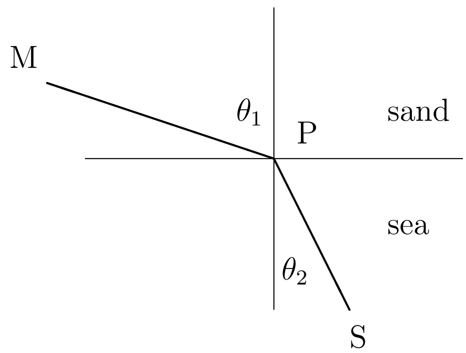

Figure 1: For Question 2(i)

(9 marks)

(ii) (a) For any symplectic form ω on $\mathbb{R}^{2n}$, define the *skew* W^σ of a vector subspace W of $\mathbb{R}^{2n}$. **(2 marks)**

(b) Using now the standard symplectic form Ω on $\mathbb{R}^{2n}$, prove that

$$\dim W^\sigma = 2n - \dim W.$$

You may use, if you wish, the Gram–Schmidt Theorem: for any $\mathbb{R}^d$ and any vector subspace W of $\mathbb{R}^d$, there is an orthogonal basis of W. **(11 marks)**

(c) Hence, or otherwise, prove that if W is a k-dimensional vector subspace of $\mathbb{R}^{2n}$ such that $\Omega(X,Y) = 0$ for all $X, Y \in W$, then $k \leqslant n$. **(3 marks)**

3 (i) Denote by $\widetilde{\mathscr{L}}$ the set of all oriented lines in $\mathbb{R}^2$. Let $U(1)$ denote the unit circle, centre the origin, and write elements of $U(1)$ as unit complex numbers $e^{i\theta}$.

(a) Define, using sketch diagrams if you wish, a bijective correspondence between $\widetilde{\mathscr{L}}$ and the product set $U(1) \times \mathbb{R}$. **(8 marks)**

(b) In terms of your correspondence in (a), find the subsets of $U(1) \times \mathbb{R}$ which correspond to

(α) the set of lines which intersect $U(1)$ in two distinct points;

(β) the set of lines which do not intersect $U(1)$;

(γ) the set of lines which are tangent to $U(1)$. **(4 marks)**

(ii) Consider the optical system shown in Figure 2. The indexes of refraction are n_1 and n_2 as shown, and $n_2 > n_1$. The boundary curves are parabolas given by $z = z_1 + \frac{1}{2}k_1q^2$ and $z = z_2 + \frac{1}{2}k_2q^2$, where $k_1 < 0$ and $k_2 > 0$ as shown.

(a) Write down the matrices which correspond to refraction at each of the parabolic boundaries, making the usual assumptions of Gaussian optics. Specify the signs of the lower-left entries in each matrix. **(4 marks)**

(b) Calculate the matrix for the optical system of Figure 2. **(4 marks)**

(c) Consider incoming horizontal rays. Show that there is no value of L for which all the outgoing rays are horizontal. **(5 marks)**

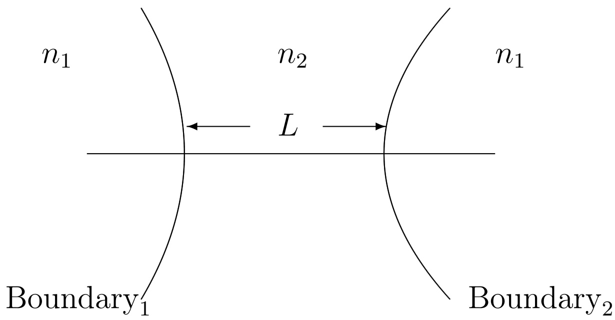

Figure 2: For Question 3(i).

4 In this question each $\mathbb{R}^{2n}$ has the standard symplectic form Ω.

(a) Let W be an n–dimensional subspace of $\mathbb{R}^{2n}$. Take a basis of W and write the elements as the columns of a $n \times 2n$ matrix which we write in block form as

$$\begin{bmatrix} M \\ N \end{bmatrix}$$

where M and N are $n \times n$ matrices. Prove that W is Lagrangian if and only if $M^{\mathsf{T}}N$ is symmetric. **(4 marks)**

(b) Let $L \subseteq \mathbb{R}^{2n}$ be a Lagrangian subspace of $\mathbb{R}^{2n}$. Show that it is transverse to both $\mathbb{R}^n \times 0$ and $0 \times \mathbb{R}^n$ if and only if it has a representation as in (a) of the form $\begin{bmatrix} M \\ I \end{bmatrix}$ with M invertible. **(10 marks)**

(c) Let L and L' be Lagrangian subspaces which are both transversal to $\mathbb{R}^n \times 0$ and $0 \times \mathbb{R}^n$. State the theorem which gives criteria for the existence of $S \in Sp(2n)$ such that $S(\mathbb{R}^n \times 0) = \mathbb{R}^n \times 0$, $S(0 \times \mathbb{R}^n) = 0 \times \mathbb{R}^n$, and $S(L) = L'$. **(3 marks)**

(d) Let

$$L = \text{span}\{(15, -6, 11, 2, 1, 0),\ (-11, 6, -11, -1, 2, 1),\ (-6, 5, -9, 0, 3, 1)\}.$$

Show that L is Lagrangian in $(\mathbb{R}^6, \Omega)$ and that it is transverse to both $\mathbb{R}^3 \times 0$ and $0 \times \mathbb{R}^3$. Represent L in the form

$$\begin{bmatrix} M \\ I \end{bmatrix}$$

as in (b) and determine the signature of M. **(8 marks)**

5 The *Local Diffeomorphism Theorem* states that if U and V are open sets in $\mathbb{R}^d$ and $\varphi\colon U \to V$ is a smooth map for which $\det D(\varphi)(Z_0) \neq 0$ for some $Z_0 \in U$, then there is an open set $U_1 \subseteq U$ with $Z_0 \in U_1$ and an open set $V_1 \subseteq V$ with $\varphi(Z_0) \in V_1$, such that the restriction $\varphi\colon U_1 \to V_1$ is a diffeomorphism.

(a) Consider a smooth function $z = F(x,y)$ defined on an open set $U \subseteq \mathbb{R}^2$. Assume that $\dfrac{\partial F}{\partial y} \neq 0$ at a particular point (x_0, y_0). Using the Local Diffeomorphism Theorem show that there is a smooth function $H(x,z)$ such that

$$F(x, H(x,z)) = z, \quad \text{and} \quad H(x, F(x,y)) = y.$$

(You need not specify any properties of the domain of H.)

Show that

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial H}{\partial x} = 0, \qquad \frac{\partial F}{\partial y}\frac{\partial H}{\partial z} = 1,$$

and

$$\frac{\partial H}{\partial x} + \frac{\partial F}{\partial x}\frac{\partial H}{\partial z} = 0.$$

(11 marks)

(b) Let $\Gamma(q,q')$ be a smooth function defined on an open set $U \subseteq \mathbb{R}^2$. Write

$$H = \frac{\partial \Gamma}{\partial q}, \qquad K = \frac{\partial \Gamma}{\partial q'}.$$

Assume that $\dfrac{\partial H}{\partial q'} \neq 0$ throughout U. Writing $p = H(q,q')$, (a) applies and shows that there is a smooth function $F(q,p)$ such that

$$H(q, F(q,p)) = p.$$

Define $G(q,p) = -K(q, F(q,p))$ and show that

$$\varphi(q,p) = (F(q,p), G(q,p))$$

is a symplectomorphism. **(8 marks)**

(c) Let $a > 0$ be a constant. Consider

$$\Gamma(q,q') = q\cos^{-1}\left(\frac{q'}{a\sqrt{2q}}\right) - \frac{q'}{a\sqrt{2}}\sqrt{q - \frac{q'^2}{2a^2}}$$

on the open set $U = \{(q,p) \mid q > 0\}$, with $\cos^{-1}$ taking values in the interval $(0,\pi)$.

Determine φ as in (b), simplifying your answer as much as possible.

(6 marks)

End of Question Paper