The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn 2011-2012**

Financial Mathematics **2 hours and 30 minutes**

Answer **three** *questions. If you answer more than three questions only your best three will be counted.*

Please leave this exam paper on your desk
Do not remove it from the hall

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to be completed by student

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1 (i) Consider the following three bonds with face value of £100:

Time to maturity (in years)	Annual interest (paid every 6 months)	Bond price (in £)
0.5	0	99.00
1.	8%	105.39
1.5	6%	104.36

(a) Find the 0.5-year spot interest rate. ***(1 mark)***

(b) Use the bootstrap method to find the 1 and 1.5-year spot interest rates. ***(9 marks)***

(c) Suppose that you are offered by a risk free institution the opportunity to deposit or borrow £1,000,000 in six months for a period of one year earning an interest rate of 4%. Describe in detail an arbitrage opportunity available to you. ***(12 marks)***

(ii) Consider a forward contract on an asset which provides no income and incurs no storage costs. Apply a no-arbitrage argument to prove that

$$F \geq Se^{rT},$$

where F is the forward price (in pounds), S is the spot price of the asset (in pounds), T is the time to maturity of the forward contract (in years) and r is the T-year spot interest rate. ***(11 marks)***

2 (i) (a) Consider a portfolio consisting of European call options on the same underlying asset and same expiration date, which is long two options with strike 10, short three options with strike 20 and long one option with strike 40. Sketch a graph of the payoff function of this portfolio. ***(4 marks)***

(b) Let c_{10}, c_{20} and c_{40} be the spot prices of the call options in part (a) with strike prices 10, 20 and 40, respectively. Let p_{40} be the spot price of a European put option on same underlying asset and same expiration date as the options above and with strike price 40. Show that $p_{40} \geq 2c_{10} - 3c_{20} + c_{40}$. ***(5 marks)***

(ii) Let c be the price of a European call option on stock which pays no dividends; let X be the strike price of the option, let S be the spot price of the stock, let T be the time to the option expiration (in years) and let r be the T-year spot interest rate. By comparing a portfolio consisting of options and cash to a portfolio consisting of one share, prove that

$$c \geq S - Xe^{-rT}.$$

(10 marks)

(iii) The price of a stock which pays no dividends is currently £100. Over each of the next three 1-year periods the stock price will either double or halve. Suppose that all interest rates are constant and equal to 10%.

(a) Use a binomial tree to find the price of a three-year American put option on this stock with strike price £150. ***(11 marks)***

(b) Describe all circumstances in which a rational investor should exercise the option. ***(3 marks)***

3 (i) (a) Explain the principle of risk-neutral valuation. ***(4 marks)***

(b) State Ito's Lemma. ***(3 marks)***

(ii) Assume that a stock price S follows the Geometric Brownian motion

$$dS = \mu S\,dt + \sigma S\,dB$$

where μ and σ are constants. Let $f = f(S,t)$ be the value at time t of a derivative contingent on the value of S at some $t = T$. Assume further that $f(s,t)$ is twice continuously differentiable with respect to s and continuously differentiable with respect to t.

(a) Show that the process followed by $f(S,t)$ is

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial f}{\partial S}\sigma S dB.$$ ***(3 marks)***

(b) Consider a portfolio consisting of a variable quantity $\frac{\partial f}{\partial S}(S,t)$ of shares and -1 derivatives; let Π be the value of this portfolio, i.e., $\Pi = \frac{\partial f}{\partial S}(S,t)S - f$. Show that after a short period of time Δt the value of the portfolio changes by

$$\Delta\Pi \approx \left(-\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 - \frac{\partial f}{\partial t}\right)\Delta t$$

(5 marks)

(c) Deduce that $\Delta\Pi \approx r\Pi\Delta t$. ***(4 marks)***

(d) Deduce that

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$ ***(2 marks)***

(iii) Let S denote the price of a stock paying no dividends. Let $c(S,t)$ and $p(S,t)$ denote the prices of European call and put options on this stock, respectively, each with strike price X and expiring in T years. Assume all interest rates are constant and equal to r.

(a) Explain why the function

$$g(S,t) = c(S,t) + Xe^{-r(T-t)} - p(S,t) - S$$

satisfies the Black-Scholes partial differential equation. ***(6 marks)***

(b) Use part (a) to deduce that $c(S,t) + Xe^{-r(T-t)} = p(S,t) + S$. ***(6 marks)***

4 (i) (a) Explain the notions of *greed*, *risk aversion* and *transitivity of preferences* in the context of portfolio theory. ***(3 marks)***

(b) Prove that two investments corresponding to the points (σ_1, r_1) and (σ_2, r_2) on the σ-r plane with $\sigma_2 > \sigma_1$ and $r_2 < r_1$ cannot lie on the same indifference curve. ***(4 marks)***

(ii) Sketch the following:

(a) an example of a feasible set and efficient frontier in the absence of a risk-free investment, ***(2 marks)***

(b) an example of a feasible set, market portfolio and efficient frontier in a market containing a risk-free investment. ***(3 marks)***

(iii) Consider risky investments A and B with the following characteristics

Investment	**Expected return**	**Standard deviation of return**
A	15%	20%
B	10%	10%

Assume also that the correlation between the returns of A and B is $\rho = 0.25$.

Find the portfolio worth £1,000,000 consisting of investments in A and B with minimal standard deviation of return and find the expected return of that portfolio. ***(6 marks)***

(iv) You are given the following data on three stocks and the market portfolio:

	Expected return	Correlation with market portfolio	Standard deviation of return
Stock 1	?	0.7	20%
Stock 2	14%	?	25%
Stock 3	4%	-0.5	?
Market portfolio	10%	1	10%

The risk-free interest rate for the period is 5%. Give the equation of the capital market line, find the beta-coefficients of Stocks 1, 2 and 3, and fill in all missing data in the table above. ***(15 marks)***

End of Question Paper