The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011-2012**

Mathematical Methods for Statistics **2 hours**

RESTRICTED OPEN BOOK EXAMINATION
Candidates may bring to the examination lecture notes and associated lecture material (including set textbooks) plus a calculator that conforms to University regulations.
Candidates should attempt ***ALL*** *questions.*
The paper will be marked out of 80 and the allocation of marks is shown in brackets.

1 Find $\displaystyle\sum_{k=2}^{n} \frac{1}{k^2-k}$, and hence find $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2-k}$. (**Hint:** partial fractions).

(10 marks)

2 (i) Differentiate $f(x) = \ln(x + x^{-1})$ with respect to x.

(ii) Let $f(x,y) = \dfrac{\cos(2xy)}{e^{-\pi(x+y)}}$. Find the partial derivative $\dfrac{\partial f}{\partial y}$.

(iii) Determine the maximum area of a rectangle with fixed perimeter p.

(10 marks)

3 Let $f(x) = \sin x$. Show that

$$f^{(n)}(\tfrac{\pi}{4}) = \begin{cases} +\dfrac{1}{\sqrt{2}} & \text{if } n = 0 + 4k \text{ or } n = 1 + 4k; \\ -\dfrac{1}{\sqrt{2}} & \text{otherwise.} \end{cases}$$

(i) Write down the Taylor's series expansion of $f(x)$ about the point $x = \frac{\pi}{4}$.

(ii) Deduce that

$$\sum_{n=0}^{\infty} \frac{\pi^n f^{(n)}(\frac{\pi}{4})}{4^n n!} = 1.$$

(10 marks)

4 Evaluate the following definite integrals, simplifying your answers as much as possible:

(i) $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin(3x)e^{\cos(3x)}dx;$

(ii) $\displaystyle\int_{\ln 2}^{\ln 3} xe^{x}dx.$ **(10 marks)**

5 Let $D = \{(x, y) \mid x^2 + y^2 \leq a^2\}$ (the disc in the xy plane of radius a centred at the origin) and $f(x, y) = h$ where h is some positive constant. By evaluating $\displaystyle\int\int_D f(x, y)dA$, find the volume of a cylinder of radius a and height h. (**Hint:** one approach is to change coordinates). **(10 marks)**

6 Let $\mathbf{x} = (1, 1, 1)$ and $\mathbf{y} = (1, 0, 1)$.

(i) Find the vector product $\mathbf{x} \times \mathbf{y}$, and verify that $(\mathbf{x} \times \mathbf{y}).\mathbf{x} = (\mathbf{x} \times \mathbf{y}).\mathbf{y} = 0$. **(5 marks)**

(ii) Show that $\{\mathbf{x}, \mathbf{y}, \mathbf{x} \times \mathbf{y}\}$ is a basis for $\mathbb{R}^3$. **(5 marks)**

7 Use Gaussian elimination to solve the following system of equations:

$$\begin{aligned} x + y + z &= 2; \\ x + 2y - z &= -1; \\ -3x + y - z &= 1. \end{aligned}$$

(10 marks)

8 Let $A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $PDP^{-1} = A$. **(10 marks)**

End of Question Paper