Data provided: Statistics Tables, H. R. Neave

MAS5051

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — **Spring Semester 2011–2012**

Probability and Probability Distributions — **2 hours**

RESTRICTED OPEN BOOK EXAMINATION.

Candidates may bring to the examination lecture notes and associated lecture material (including set textbooks) plus a calculator that conforms to University regulations.

Candidates should attempt ***All*** *questions.*

The maximum marks for the various parts of the questions are indicated.

The paper will be marked out of 80.

PDF stands for Probability Density function and CDF for Cumulative Distribution Function.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

Blank

1 (i) Let T be a random variable distributed according to a t distribution with 14 degrees of freedom.

(a) Find the value of y such that $\mathrm{P}[T \le y] = 0.075$. *(2 marks)*

(b) Use tables to find bounds for $\mathrm{P}[T > 3.5]$. *(2 marks)*

(ii) Let F be distributed according to an F distribution with 6 and 6 degrees of freedom.

(a) Use tables to find bounds for $\mathrm{P}[F > 5]$. *(2 marks)*

(b) Use tables to find bounds for $\mathrm{P}[F < 0.15]$. *(2 marks)*

2 Let X and Y be random variables with joint PDF

$$f(x,y) = \begin{cases} k\left(x^2 + y\right) & (x,y) \in S \\ 0 & \text{otherwise} \end{cases}$$

with S, the unit square

$$S = \{(x,y) \,:\, 0 \le x \le 1, 0 \le y \le 1\} \;.$$

(i) Find the value k that makes $f(x,y)$ a valid PDF. *(3 marks)*

(ii) Find the marginal distributions of X and Y. Are X and Y independent? *(5 marks)*

(iii) Calculate $\mathrm{P}[X + Y > 1]$. *(6 marks)*

3 Assume that X is continuous with PDF

$$f(x) = \alpha\beta^{\alpha}x^{-(\alpha+1)} \;; \qquad x \ge \beta, \quad \alpha, \beta > 0.$$

(i) Sketch $f(x)$, for $\beta = 2$ and $\alpha = 2$. *(3 marks)*

(ii) Show that $U = 1 - (\beta/X)^{\alpha}$ follows a Uniform distribution on $(0,1)$. *(6 marks)*

(iii) Write down the variance of U. *(2 marks)*

(iv) Suppose $\beta = 1$. Calculate $\mathrm{P}[X > 3]$ if

(a) $\alpha = 2$. *(3 marks)*

(b) $\alpha = 5$. *(3 marks)*

4 Let $\boldsymbol{X} = (X_1, X_2, X_3)$ be a random vector which follows a multivariate Normal distribution, with mean $\boldsymbol{\mu} = (0, 2, 1)$ and covariance matrix

$$\Sigma = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}$$

(i) Calculate the correlation between X_2 and X_3 *(3 marks)*

(ii) Let $\boldsymbol{Y} = A\boldsymbol{X}$, with

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

(a) Specify completely the joint distribution of $\boldsymbol{Y}$. *(7 marks)*

(b) What does the correlation matrix of $\boldsymbol{Y}$ tell you about Y_1, Y_2 and Y_3? *(3 marks)*

(c) Use the marginal distribution of Y_3 to calculate $\mathrm{P}[X_3 > X_2]$. *(3 marks)*

5 You are offered the chance to enter the following game: One of two dice with their sides marked 1,2,3,4,5,6 is selected at random. One of them, say dice A, is fair; while the other, say dice B, lands on each of the odd numbered faces with probability $1/12$ and on each of the even ones with probability $1/4$.

Once a dice is selected, it is rolled two times and the sum of the numbers showed, S_1, recorded. Then the dice is thrown twice again and the new sum, S_2 recorded. You win the game if both sums are 7, *i.e.* if $(S_1, S_2) = (7, 7)$.

(i) Calculate the probability of winning if

(a) dice A is selected, *(3 marks)*

(b) dice B is selected. *(3 marks)*

(ii) You decide to enter the game and win. Calculate the probability that you played with the fair dice, A, if

(a) both dice were equally likely to be chosen, *(3 marks)*

(b) dice B was selected with probability $3/4$. *(3 marks)*

(iii) Calculate the prior probability of selecting dice A that would make the posterior probabilities of selecting either dice equal given a win. *(5 marks)*

6 An individual player scoring three goals in a single football match is so rare that it even has a name: a "hat trick". Using the historical records for a particular league, there is on average a hat trick every 10 matches.

(i) Considering that there are 30 teams in that championship, making appropriate assumptions calculate the probability of seeing 3 or more hat tricks in any given match day. ***(4 marks)***

(ii) The tournament ends when each of the 30 teams has played twice against every other team, *i.e.* each team plays 58 matches in the tournament. Give an interval, $[a, b]$ such that the number of hat tricks in the tournament is in $[a, b]$ with probability 0.95 approximately. ***(4 marks)***

End of Question Paper