The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2011–2012**

Introductory Mathematical Finance and Time Series **3 hours**

Marks will be awarded for your best ***five*** *answers.*

CLOSED BOOK EXAMINATION for Questions 1-3 and RESTRICTED OPEN BOOK EXAMINATION for Questions 4-6.
For the Open book examination part, candidates may bring to the examination lecture notes and associated lecture material (but no textbooks) plus a calculator that conforms to University regulations.
There are 100 marks available on the paper.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) Consider the following three bonds with face value of £100:

Time to maturity (in years)	Annual interest (paid every 6 months)	Bond price (in £)
0.5	0	99.00
1.	8%	105.39
1.5	6%	104.36

(a) Find the 0.5-year spot interest rate. ***(1 mark)***

(b) Use the bootstrap method to find the 1 and 1.5-year spot interest rates. ***(8 marks)***

(c) Suppose that you are offered by a risk free institution the opportunity to deposit or borrow £1,000,000 in six months for a period of one year earning an interest rate of 4%. Describe in detail an arbitrage opportunity available to you. ***(11 marks)***

2 (i) (a) Consider a portfolio consisting of European call options on the same underlying asset and same expiration date, which is long two options with strike 10, short three options with strike 20 and long one option with strike 40. Sketch a graph of the payoff function of this portfolio. ***(4 marks)***

(b) Let c_{10}, c_{20} and c_{40} be the spot prices of the call options in part (a) with strike prices 10, 20 and 40, respectively. Let p_{40} be the spot price of a European put option on same underlying asset and same expiration date as the options above and with strike price 40. Show that $p_{40} \geq 2c_{10} - 3c_{20} + c_{40}$. ***(5 marks)***

(ii) Let S denote the price of a stock paying no dividends. Let $c(S,t)$ and $p(S,t)$ denote the prices of European call and put options on this stock, respectively, each with strike price X and expiring in T years. Assume all interest rates are constant and equal to r.

(a) Explain why the function

$$g(S,t) = c(S,t) + Xe^{-r(T-t)} - p(S,t) - S$$

satisfies the Black-Scholes partial differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf. \qquad \textit{(5 marks)}$$

(b) Use part (a) to deduce that $c(S,t) + Xe^{-r(T-t)} = p(S,t) + S$. ***(6 marks)***

3 (i) Sketch the following:

(a) an example of a feasible set and efficient frontier in the absence of a risk-free investment, ***(2 marks)***

(b) an example of a feasible set, market portfolio and efficient frontier in a market containing a risk-free investment. ***(3 marks)***

(ii) You are given the following data on three stocks and the market portfolio:

	Expected return	Correlation with market portfolio	Standard deviation of return
Stock 1	?	0.7	20%
Stock 2	14%	?	25%
Stock 3	4%	-0.5	?
Market portfolio	10%	1	10%

The risk-free interest rate for the period is 5%. Assume that the Capital Asset Pricing Model holds. Give the equation of the capital market line, find the beta-coefficients of Stocks 1, 2 and 3, and fill in all missing data in the table above. ***(15 marks)***

4 (i) The figure below shows the quarterly averages of the Euro/£ and US$/£ exchange rates between 2001 and 2005 (source: Bank of England). Also plotted is the price of gold relative to its price in the first quarter of 2001. Describe the three time series and their relationship, using suitable technical terms and adding approximate quantification where appropriate. Detailed numerical comparisons of the series are not required. ***(4 marks)***

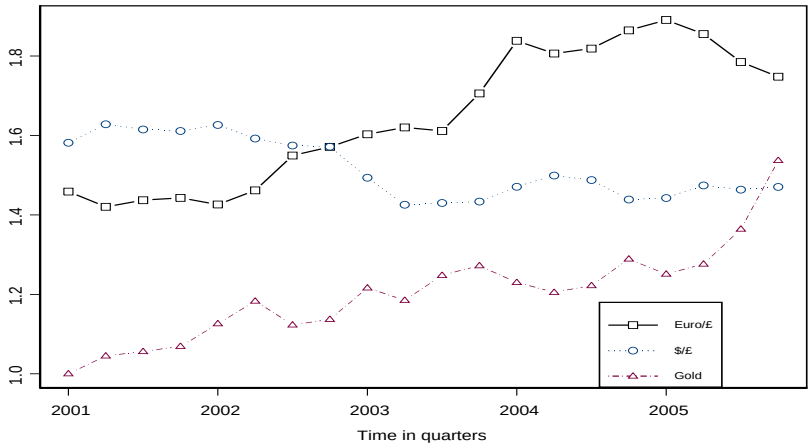

Figure 1: Dollar/Pound, Euro/Pound and Gold Index, Jan 2001 - Dec 2005

(ii) The following table shows the sample acf r_h and pacf $a_h^{(h)}$ values of the Dollar/pound series where the lag $h = 1, 2, 3, \ldots$ is in quarters of a year.

Lag h	1	2	3	4	5	6	7	8
r_h	0.9123	*	0.6287	0.4604	0.2703	0.0898	-0.0653	-0.2012
$a_h^{(h)}$	*	-0.3586	-0.0071	-0.2779	-0.1745	-0.0563	-0.0252	-0.0706

Calculate the two missing values. Without making any further calculations, describe briefly how using the above table, you could investigate whether the time series is stationary or not. ***(7 marks)***

(iii) Investigate and compare possible models for the series given the values in the table in (ii) and your calculations for the missing ones. What further evidence is desirable for a more complete identification of a suitable model? ***(9 marks)***

5 Consider the time series model

$$X_t = \frac{1}{3}X_{t-1} + \epsilon_t + \frac{1}{2}\epsilon_{t-1} - \frac{1}{4}\epsilon_{t-2},$$

where ϵ_t is white noise with variance 4.

(i) Explain why in this model X_t has zero mean. ***(2 marks)***

(ii) Show that this model is invertible. ***(3 marks)***

(iii) If $Var(X_t) = 9$, calculate the autocorrelation function (ACF) of X_t. ***(10 marks)***

(iv) Find a state space representation for the model for X_t. Write down the observation and evolution equations and state the distribution of the observation and evolution innovations. ***(5 marks)***

6 (i) Consider the AR(1) time series model

$$X_t = \alpha X_{t-1} + \epsilon_t,$$

where ϵ_t is a white noise sequence with variance σ^2.

(a) Derive the formula

$$X_{n+k} = \alpha^k X_n + \sum_{i=0}^{k-1} \alpha^i \epsilon_{n-i}$$

for some positive integers n and k. ***(2 marks)***

(b) Using (a) or otherwise, derive the k-step ahead forecast mean and variance of X_{n+k}, based on observed time series data $X_1 = x_1, \ldots, X_n = x_n$. ***(7 marks)***

(c) If ϵ_t is normally distributed, write down a 95% prediction interval for X_{n+k}. ***(1 mark)***

(ii) Consider the AR(2) time series model

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \epsilon_t,$$

where ϵ_t is as above a white noise sequence with variance σ^2.

(a) Write down X_t in state-space form. ***(1 mark)***

(b) Using the state-space form in (a), derive the 2-step ahead forecast mean and variance of X_{n+2}, based on observed time series data $X_1 = x_1, \ldots, X_n = x_n$. ***(7 marks)***

(c) Hence show that the 2-step forecast variance of X_{n+2} is the same as that of an AR(1) model defined by $X_t = \alpha_1 X_{t-1} + \epsilon_t$. ***(2 marks)***

End of Question Paper