Data provided: Formulae sheet

The University Of Sheffield.

CIV340

SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester 2012-2013

Computational Engineering Mathematics

Three hours

Marks will be awarded for your best FOUR answers

1 (i) Consider the following convection-diffusion equation for temperature distribution $u(x,t)$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \quad (0 \le x \le 1). \tag{1}$$

Using the notation $u_{ij} \equiv u(x_i, t_j)$ and the conventions that $i = 0$ and $i = N$ correspond to $x = 0$ and $x = 1$, respectively, and $j = 0$ corresponds to $t = 0$, derive the implicit scheme

$$-(k - \beta)u_{i+1,j} + (1 + 2k)u_{ij} - (k + \beta)u_{i-1,j} = u_{i,j-1}$$

for $i = 1, .., N-1,\ \ j = 1, 2, ..,$ where $k = \Delta t/(\Delta x)^2$ and $\beta = \Delta t/(2\Delta x)$. Use suitable finite difference approximations given on the formulae sheet.

(5 marks)

(ii) Equation (1) is to be solved (approximately) over the range $0 \le x \le 1$ with conditions $u(0,t) = 20$ and $u(1,t) = 10$ and initial temperature distribution $u(x,0) = 20 - 10x$. Assuming $\Delta x = 0.25$ and $\Delta t = 0.2$, use the above implicit scheme to write down the system of algebraic equations for the temperature at $x = 0.25, 0.5, 0.75$ and time $t = 0.2$. ***(12 marks)***

(iii) Derive the Jacobi iteration equation to find the $(k+1)$th iteration, from the kth iteration, for the unknowns in the system of algebraic equations derived above. Use the temperature values at $t = 0$ as the approximate solutions at the kth iteration, find the solution at the $(k+1)$th iteration.

(8 marks)

2 You are given the relation

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

where σ_{ij} is the stress tensor, ε_{ij} is the (symmetric) strain tensor, and C_{ijkl} is a fourth order coefficient tensor. In what follows δ_{ij} is the Kronecker delta tensor.

(i) Given that $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, show that

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{mm} + 2\mu\varepsilon_{ij}.$$

Then write down the $(1, 1)$ component of the above equation. ***(8 marks)***

(ii) Show that the mean normal stress $\sigma_{kk}/3$ is

$$\frac{1}{3}\sigma_{kk} = (\lambda + \frac{2}{3}\mu)\varepsilon_{mm}.$$

(3 marks)

(iii) Given $\lambda = 0.7790$ and $\mu = 1.4092$, and that the strain tensor is defined by $\varepsilon_{11} = 0.7013$, $\varepsilon_{22} = -0.31\varepsilon_{11}$, $\varepsilon_{33} = -0.19\varepsilon_{11}$, $\varepsilon_{12} = 0.2274$, $\varepsilon_{23} = 0.6413$, and $\varepsilon_{31} = -0.8273$, find the stress tensor, and then the stress force on a unit area element in the direction $\mathbf{n} = (0.7071, -0.7071, 0)^T$. ***(14 marks)***

3 (i) Consider a small Δx by Δy material element in a thin plate. The state of stress in the plate is specified by four components σ_{xx}, σ_{yy} and $\sigma_{xy} = \sigma_{yx}$, and the state of strain is specified by ε_{xx}, ε_{yy} and $\varepsilon_{xy} = \varepsilon_{yx}$. The other components are zero. Let the x-component of the body force acting on the body be F_x, show that the equation of equilibrium for the element in the x-direction is

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + F_x = 0.$$

(14 marks)

(ii) Given the constitutive relation between the stress tensor and the strain tensor

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{mm} + 2\mu \varepsilon_{ij},$$

and that the displacement of the point (x, y) in the plate is given by $u(x, y)$ in the x-direction and $v(x, y)$ in the y-direction, show that the equation of equilibrium derived above can be written as

$$\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + (\lambda + \mu) \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + F_x = 0.$$

(11 marks)

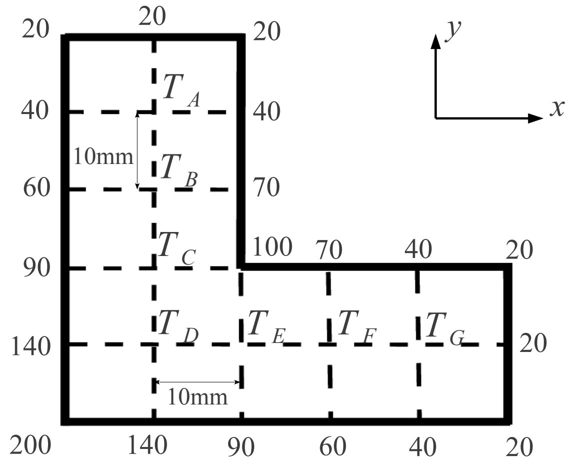

Figure Q4a: An L-shape plate with temperature defined on the boundaries.

4 (i) The temperature distribution in the L-shape plate shown in Figure Q4a satisfies the indicated boundary conditions (given in degrees centigrade) and the Poisson's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -20.$$

(a) Draw the solution domain showing the line of symmetry for the temperature distribution and indicate which of the unknown temperatures are equal to each other. ***(4 marks)***

(b) Use the formulae on the formulae sheet to formulate the finite difference equations required to find the approximate solutions of the temperatures T_A, T_B, T_C and T_D. ***(13 marks)***

 Question 4 continued on next page

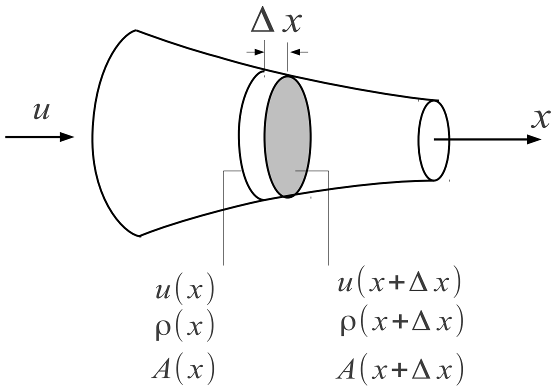

Figure Q4b: The fluid motion in a pipe.

4 (continued)

(ii) Consider the fluid motion in a pipe with varying area of cross-section shown in Figure Q4b. The velocity and the density of the fluid are uniform in each cross section. Let the x-axis point in the axial direction of the pipe, and let $A(x)$ be the area of the cross section, $\rho(x)$ the density and $u(x)$ the velocity at x. By considering the mass of the fluid in a fixed volume between the cross-sections at x and $x + \Delta x$, derive the following continuity equation based on conservation of mass:

$$\frac{\partial \rho A}{\partial t} + \frac{\partial \rho u A}{\partial x} = 0.$$

Hint: the mass of fluid passing through the cross section A in a unit time is $\rho u A$. ***(8 marks)***

5 (i) Given the equation and the boundary conditions

$$\frac{d^2u}{dx^2} - 9u - 3x^2 = 0, \quad u(0) = 0 \text{ and } u(1) = 1,$$

for unknown function $u(x)$, show that the weak form of the equation is

$$\int_0^1 \frac{du}{dx}\frac{dw}{dx}dx + 9\int_0^1 uwdx = w(1)u'(1) - w(0)u'(0) - 3\int_0^1 x^2 wdx,$$

where $u'(x) = du/dx$ and $w(x)$ is the weight function. ***(7 marks)***

(ii) Let the trial solution be

$$u(x) = x + c_1x(1-x) + c_2x(1-x^2),$$

where c_1 and c_2 are constants to be determined. Choose $w(x) = x$ and $w(x) = x^2$, respectively, derive the equations for c_1 and c_2 using the weak form equation, and then solve for c_1 and c_2. ***(18 marks)***

End of Question Paper

Formulae Sheet

Notation:

$$U(x_i,\ t_j) \equiv U_{i\,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1} - U_{ij}}{\Delta t}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{ij} - U_{i,j-1}}{\Delta t}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{\Delta x^2}$$