The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2012–2013

FOUNDATION YEAR MATHEMATICS II — 3 hours

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) (a) Draw the triangle with side lengths 1, $\sqrt{3}$, and 2. Your diagram should include the angles of the triangle given in both radians and degrees. ***(4 marks)***

(b) You are given that $90^\circ \leq A \leq 180^\circ$ and $\tan(A) = -\frac{4}{3}$, find values for $\sin(A)$ and $\cos(A)$. ***(4 marks)***

(ii) Let $\mathbf{a} = 3\mathbf{i} - 4\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \sqrt{2}\mathbf{j} + \mathbf{k}$.

(a) Express $\mathbf{a}$, $\mathbf{b}$ in the form (x, y, z) for real numbers $x, y, z \in \mathbb{R}$. ***(2 marks)***

(b) Compute the angle between $\mathbf{a}$ and $\mathbf{b}$. ***(3 marks)***

(c) Write down a vector that is at right angles to both $\mathbf{a}$ and $\mathbf{b}$. ***(3 marks)***

(iii) A particle moves through $\mathbb{R}^3$ in a straight line in the $\mathbf{i} + \mathbf{j} + \mathbf{k}$ direction at a constant speed of 3 unit/sec. At time $t = 0$ the particle is at the point $(1, 0, 0)$. At what time does the particle pass through $(1 + 3\sqrt{3}, 3\sqrt{3}, 3\sqrt{3})$? ***(4 marks)***

2 (i) State the fundamental periods of $\sin(x)$, $\cos(2x)$, and $\tan(\frac{x}{2}+\frac{\pi}{2})$ in radians. ***(6 marks)***

(ii) Express $\tan(x)$ in terms of $\sin(x)$ and $\cos(x)$. Hence or otherwise show that

$$2\sin(x)\cos(x) = \frac{2\tan(x)}{1+\tan^2(x)}.$$

(7 marks)

(iii) Find *all* solutions to

$$\cos(x) = \frac{1}{\sqrt{2}},$$

and, for $\frac{\pi}{2} \le x \le \frac{5\pi}{2}$, find *all* solutions to

$$\cos^4(x) - \frac{1}{4} = 0.$$

(7 marks)

3 (i) Find

(a) $1+2+3+4+5+6+\cdots+100$.

(b) $\sum_{n=1}^{\infty}(\frac{1}{2})^n$. ***(8 marks)***

(ii) A deck of cards has 52 cards, and is made up of four suits. Each suit has 13 cards. In how many ways can two distinct cards of the same suit be drawn from a deck of cards? ***(4 marks)***

(iii) State the binomial expansion of $(x+y)^n$ for a positive integer n. ***(4 marks)***

(iv) By setting $n = 100$ and choosing appropriate values for x and y in the binomial expansion of $(x+y)^n$, find

$$\sum_{k=0}^{100}\binom{100}{k}(-1)^k.$$

(4 marks)

4 (i) Write down a parametric equation for the line described by

$$y = x + 1.$$

(3 marks)

(ii) Express the circle C in the form $(x - a)^2 + (y - b)^2 = r^2$ where C is parametrised by

$$C = \{(2\cos(\theta) + 1, 2\sin(\theta) - 1) : \theta \in [0, 2\pi)\}$$

(3 marks)

(iii) Let $w_1 = 1 + i$ and $w_2 = -1 + i$. Compute the modulus of w_1 and the argument of w_2, and express $\overline{w}_1$, $w_1 w_2$, and $\frac{w_1}{w_2}$ in the form $a + bi$ where $a, b \in \mathbb{R}$.

(10 marks)

(iv) Draw an Argand diagram indicating the set $\{z : |z - i| = 1\}$. ***(4 marks)***

5 (i) Use the Newton-Raphson method of approximation with $x_1 = -1$ to approximate a solution to $f(x) = 0$, where $f(x) = 4x^3 - 2x + 1$. You should iterate the process two times; so your final answer should be x_3. ***(6 marks)***

(ii) The equation of the upper unit semi-circle centred at the origin is given by:

$$y = +\sqrt{1 - x^2}.$$

Using the five ordinates $x = \pm 1$, $x = \pm\frac{1}{2}$, and $x = 0$, approximate the area of the region R bounded by the x-axis and the upper unit semi-circle using

(a) the trapezium method,

(b) Simpson's rule.

Use your answers to (a) and (b) to determine which method gives the better approximation to the area of R. ***(10 marks)***

(iii) Suppose that a function $x(t)$ satisfies $x'(t) = 2x(t)$. You are given that at time $t = 0$ the function $x(t) = 10$. At what time does $x(t) = 1000$? ***(4 marks)***

End of Question Paper