The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2012–13**

Mathematics Core I **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

Total marks 60.

1 Let $f : \mathbb{R} \to \mathbb{R}$ be the function such that $f(x) = e^{-x^2}$.

(i) Calculate the derivative $f'(x)$, and the second derivative $f''(x)$. Find the values of x for which $f'(x) = 0$, and those for which $f''(x) = 0$. Sketch the graph of f, marking on it the points corresponding to the x-values you just found. ***(7 marks)***

(ii) Give an example to show that f is not an injective function. What is the image (or range) of f? ***(2 marks)***

2 (i) Let $A = \{1, 2, 3, 4\}$, $B = \{4, 5, 6\}$ and $C = \{2, 4, 6\}$. Write down (by listing their elements between curly brackets) the sets $A \cap C$, $A \cup C$ and $(A \cap C) \times B$ ***(3 marks)***

(ii) How many functions $f : \{1, 2, 3\} \to \{1, 2, 3, 4\}$ are there? How many are injective? How many are surjective? ***(3 marks)***

3 Starting from $y + \Delta y = (x + \Delta x)^3$, prove that if $y = x^3$ then $\dfrac{dy}{dx} = 3x^2$. ***(3 marks)***

4 (i) Consider the function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ given by $g(t) = e^{-t}\cos t$. For what values of t is $g(t) = 0$? Calculate $g'(t)$ and $g''(t)$. Sketch the graph of g. Using your results on $g'(t)$ and $g''(t)$, find a second-order differential equation, of the form $y'' + ay' + by = 0$, for which a solution is $y = g(t)$. Write down the general solution, and find the particular solution such that $y(0) = 1$ and $y'(0) = 2$. ***(8 marks)***

(ii) Let $G : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be defined by $G(x) = \int_0^x e^{-t}\cos t\,dt$. On your graph for part (i), mark an area naturally representing $G(\pi/4)$. Without evaluating the integral, what is $G'(x)$? Explain briefly why $G(x)$ takes its maximum value at $x = \pi/2$. Calculate that maximum value. Write down the value of x for which G takes its minimum value. ***(7 marks)***

5 Find the solution for $y' + xy = e^{-x^2/2}$ such that $y(0) = 2$ ***(3 marks)***

6 (i) Write down the formulas for $\sin(\theta+\phi)$ and $\cos(\theta+\phi)$ in terms of $\sin\theta, \cos\theta$, $\sin\phi$ and $\cos\phi$. *Deduce* formulas for $\sin(2\theta)$ and $\cos(2\theta)$. Using the latter, *deduce* formulas for $\sin^2\theta$ and $\cos^2\theta$. ***(3 marks)***

(ii) Using the substitution $x = 2\sin\theta$ (strictly speaking, $\theta = \sin^{-1}(x/2)$), evaluate the integral $\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$. ***(6 marks)***

7 (i) If $\sum_{n=0}^{\infty} a_n x^n$ is the Maclaurin series for f, write down the formula for a_n. Use it to determine the Maclaurin series for $\sin x$. ***(3 marks)***

(ii) You may assume that, for all $x \in \mathbb{R}$, $\sin x$ is the sum of the series found in part (i). Hence or otherwise, explain why $\sin(1/2) < 1/2$. ***(1 mark)***

(iii) What is $\sin(\pi/6)$? Using part (ii), deduce that $\pi > 3$. ***(1 mark)***

8 (i) Let $z = 2+3i$ and $w = 1-i$. Calculate $z+\overline{w}$, zw and $z/\overline{w}$. Express w in the form $re^{i\theta}$, with $r \geq 0$ and $-\pi < \theta \leq \pi$. ***(4 marks)***

(ii) State Euler's identity for $e^{i\theta}$. By raising to the n^{th} power, deduce de Moivre's theorem. Letting $n = 6$, obtain expressions for $\cos 6\theta$ and $\sin 6\theta$ purely in terms of $\cos\theta$ and $\sin\theta$. (Both may appear in each formula.) Deduce a formula for $\tan 6\theta$ purely in terms of $\tan\theta$. What happens if you plug in $\theta = \pi/4$ to this formula for $\tan 6\theta$? ***(6 marks)***

End of Question Paper