The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Spring Semester 2012–2013**

**Mathematics Core II** **2 hours**

*Attempt all the questions. The allocation of marks is shown in brackets.*

## Section A

**A1** Solve the following inequalities:

(i) $|2x-5|<7$; ***(2 marks)***

(ii) $|x^2+4x-5|>0$. ***(2 marks)***

**A2** Evaluate the following limits:

(i) $\displaystyle\lim_{x\to 1}\frac{x^2+4x-5}{x^2-3x+2}$; ***(2 marks)***

(ii) $\displaystyle\lim_{x\to 2\pi}\frac{x^2-\pi x-2\pi^2}{\sin 4x}$. ***(2 marks)***

**A3** Give the equation of the tangent plane to the surface $z=4x^2+xy+y^2$ at the point $(1,1,6)$. ***(3 marks)***

**A4** Let $f(x,y)=\dfrac{x+y}{\sqrt{x^2-y^2}}$. Evaluate $f$ at the point $(5,3)$. Using partial derivatives, approximate the change $\delta f$ when we move to the nearby point $(4.9,3.1)$. Compare this approximation with the actual value of $\delta f$. ***(5 marks)***

**A5** Let $z$ be a function of $u$ and $v$ where $u = x + y$ and $v = 3x - 3y$.

(i) Show that

$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = \left(\frac{\partial z}{\partial u}\right)^2 - 9\left(\frac{\partial z}{\partial v}\right)^2.$$

***(2 marks)***

(ii) Assuming equality of mixed second-order partial derivatives, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 2\frac{\partial^2 z}{\partial u^2} + 18\frac{\partial^2 z}{\partial v^2}.$$

***(3 marks)***

**A6** Calculate the integral of the function $f(x, y) = xy$ over the region bounded by the circle of radius 1 centred on the origin, the $y$-axis and the line $y = \frac{1}{2}$. ***(5 marks)***

**A7** (i) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + n^3}$ converge? Briefly justify your answer.

***(2 marks)***

(ii) What is the radius of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(4n)!}{(n!)^4 2^n} z^n$ ? ***(2 marks)***

## Section B

**B1** Solve the following system of linear equations.

$$\begin{cases} x + y + 3z + w = 2, \\ x - y + z + w = 4, \\ y + 2z + 2w = 0. \end{cases}$$

***(4 marks)***

**B2** Use the determinant test to investigate whether the following vectors

$$\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix},$$

are linearly dependent or linearly independent. ***(2 marks)***

**B3** Find an inverse of the matrix $A$ by using the Gauss-Jordan elimination on

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 \end{array}\right),$$

where $I$ denotes an identity matrix. ***(4 marks)***

**B4** Determine the eigenvalues and the corresponding eigenvectors for

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

***(6 marks)***

**B5** (i) State the focus and the directrix property of an ellipse and derive its standard equation on this basis. ***(4 marks)***

(ii) Show that for all $m (\neq 0)$ the line

$$y = mx + \frac{a}{m}$$

is tangent to the parabola

$$y^2 = 4ax$$

and determine the point of tangency. ***(4 marks)***

**B6** Consider three circles on the plane:

$$C_1 : x^2 + y^2 - 2ax + 1 = 0,$$

$$C_2 : x^2 + y^2 - 2by - 1 = 0,$$

$$C_3 : x^2 + y^2 - ax - by = 0,$$

where $a > 1,\ b > 0$.

(i) Show that $C_3$ passes through the centres of $C_1$ and $C_2$. ***(2 marks)***

(ii) Show that $C_1$ and $C_2$ intersect at two distinct points. (These points will be called $P_1$ and $P_2$ below.) ***(2 marks)***

(iii) Show that $C_3$ passes through $P_1$ and $P_2$. ***(2 marks)***

**End of Question Paper**