The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012-2013**

Vectors and Mechanics

2 **Hours**

Attempt all the questions. The allocation of marks is shown in brackets.
The total number of marks available is 60.

1 Two points, A and B, have position vectors $\boldsymbol{a} = 4\boldsymbol{i} - 8\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 7\boldsymbol{i} + \boldsymbol{j}$ respectively.

Find

(i) The vector $\overrightarrow{AB}$;

(ii) The position vector of the mid-point of AB;

(iii) The vector equation of the line AB.

(4 marks)

2 Find a vector parallel to the line L with vector equation

$$\boldsymbol{r} = 2\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k} + \lambda(\boldsymbol{i} - \boldsymbol{j}).$$

Find a vector perpendicular to the plane Π with vector equation

$$\boldsymbol{r}.\boldsymbol{k} = 6.$$

Explain why the line L is parallel to the plane Π.

Explain why the line L lies in the plane Π. ***(5 marks)***

3 Given the vectors

$$\boldsymbol{u} = 2\boldsymbol{i} - 3\boldsymbol{j} + 8\boldsymbol{k}, \qquad \boldsymbol{v} = 6\boldsymbol{i} - 4\boldsymbol{j} - \boldsymbol{k},$$

find $\boldsymbol{u}.\boldsymbol{v}$, $\boldsymbol{u} \times \boldsymbol{v}$ and the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$, giving your answer in radians correct to three significant figures. ***(5 marks)***

4 A constant force $\boldsymbol{F} = [6\boldsymbol{i} - 36\boldsymbol{j} + 54\boldsymbol{k}]$ N acts on a particle of mass 3 kg.
At time $t = 0$ s, the particle is at the point with position vector $[\boldsymbol{i} - 5\boldsymbol{j} - \boldsymbol{k}]$ m.
At time $t = 1$ s, the particle is at the point with position vector $[5\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}]$ m.

Find

(i) The acceleration of the particle at time t s.

(ii) The work done by the force between time $t = 0$ s and time $t = 1$ s.

(iii) The impulse of the force between time $t = 0$ s and time $t = 1$ s.

(4 marks)

5 A particle of mass m moves in such a way that its position vector at time t is given by

$$\boldsymbol{r} = a\boldsymbol{i}\cos(\omega t) + b\boldsymbol{j}\sin(\omega t),$$

where a, b and ω are positive constants.

(i) Find the momentum $\boldsymbol{p}$ of the particle at time t.

(ii) Show that

$$\boldsymbol{r}.\boldsymbol{p} = \frac{1}{2}m\omega\left(b^2 - a^2\right)\sin(2\omega t).$$

(iii) Show that

$$\boldsymbol{r} \times \boldsymbol{p} = mab\omega\boldsymbol{k}.$$

(iv) What are the dimensions of the constants a, b and ω?

(6 marks)

6 A truck of mass M has an engine which generates power P. The truck is travelling in a straight line along a rough horizontal track. The coefficient of friction between the track and the truck is μ.

Draw a clear diagram showing all the forces on the truck.

Find the maximum speed U at which the truck can move along the track in terms of P, M, μ and the acceleration due to gravity g.

The truck is travelling at its maximum speed U when the engine is switched off. Find the time T taken for the truck to come to rest, in terms of P, μ, g and M. ***(6 marks)***

7 A particle of mass 3 kg is suspended from one end of an elastic string of natural length 0.5 m and stiffness 98 N m^{-1}. The other end of the string is attached to a fixed point O. In this question, use the value $g = 9.8$ m s^{-2} for the acceleration due to gravity and give all answers correct to three significant figures. Air resistance can be ignored.

(i) Draw a clear diagram showing all the forces acting on the particle.

(ii) Find the extension of the string when the particle is in equilibrium.

(iii) If the particle is pulled vertically downwards and released when the length of the string is 1 m, find the acceleration of the particle the instant it is released.

(iv) Find the period of the subsequent oscillations of the particle.

(10 marks)

8 (i) A catapult projects a stone from the surface of a horizontal playground. The stone is projected with velocity V at an acute angle α to the playground. Air resistance can be ignored.

Show that the time after projection at which the stone hits the playground is

$$t = \frac{2V}{g} \sin \alpha$$

where g is the acceleration due to gravity.

(ii) Suppose now the playground is inclined at an angle β to the horizontal.

The catapult on the playground now projects the stone with velocity V at an acute angle α to the playground, in the vertical plane through the line of greatest slope. Air resistance can be ignored.

Show that the time after projection at which the stone hits the playground in this case is

$$t = \frac{2V \sin \alpha}{g \cos \beta}.$$

(8 marks)

9 A particle P of mass m moves on the surface of a smooth sphere of radius R. The line joining the particle to O, the centre of the sphere, makes an angle $\theta(t)$ with the *upwards* vertical.

The particle moves on a vertical circle on the surface of the sphere. The acceleration of the particle is given by

$$\ddot{\boldsymbol{r}} = R\ddot{\theta}\boldsymbol{e}_\theta - R\dot{\theta}^2\boldsymbol{e}_r.$$

(i) Draw a clear diagram showing all the forces on the particle and the directions of the unit vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$.

(ii) Show that the normal contact force N on the particle is given by

$$N = mg\cos\theta - mR\dot{\theta}^2.$$

(iii) The particle is released from rest at the top of the sphere.
Find the value of $\cos\theta$ when the particle leaves the surface of the sphere.

(12 marks)

End of Question Paper