



The
University
Of
Sheffield.

MAS151

SCHOOL OF MATHEMATICS AND STATISTICS

**Spring Semester
2012–2013**

MAS151 Civil Engineering Mathematics

3 hours

Attempt ALL questions.

*Each question in Section A carries 3 marks,
each question in Section B carries 8 marks.*

Section A

A1 Use the binomial theorem to find the value of a for which $\lim_{x \rightarrow 0} \frac{(8 - ax)^{1/3} - 2}{x} = 1$.

A2 If $f(x) = \frac{3x}{x+2}$, $x \geq 0$, what is the range of $f(x)$? Find $f^{-1}(y)$.

A3 Find the first three nonzero terms in the Maclaurin expansion of $f(x) = e^{-2x} \cos 3x$.

A4 Evaluate $\lim_{x \rightarrow \pi/2} \frac{x \cos 3x}{2x - \pi}$.

A5 Find all the complex numbers z for which $\operatorname{Re}(z) + \operatorname{Im}(z) = 0$ and $|z - 1 + i| = \sqrt{2}$.

A6 If $\mathbf{a} = (3, 1, -1)$ and $\mathbf{b} = (1, -2, 1)$, show that \mathbf{a} is perpendicular to \mathbf{b} and find $\mathbf{a} \times \mathbf{b}$.

A7 Use integration by parts to find the indefinite integral

$$\int x^n \ln(x) dx$$

where $n \neq -1$.

A8 Compute the definite integral

$$\int_{-1}^1 \sin^2\left(\frac{\pi x}{2}\right) dx.$$

A9 Find the determinant of $A^{-1}B$, where A and B are given by

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}.$$

A10 Consider the homogeneous linear system of equations

$$\begin{aligned} 3x + y &= 0, \\ ax - 4y &= 0, \end{aligned}$$

which has the trivial solution $x = y = 0$. Find the value of a for which the system has additional solutions.

A11 Find the general solution of the following differential equation:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0.$$

A12 Show that the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{1-x^2},$$

is given by $y = A\sqrt{\frac{1+x}{1-x}}$, where A is a constant.

Section B

- B1** Find the quadratic $Q(x)$ for which the polynomial $P(x) = x^3 - x^2 - x - 15$ can be written in the form

$$P(x) = (x - 3)Q(x).$$

Find the roots of $Q(x)$ and plot them on an Argand diagram. Find the constants A and B such that

$$Q(x) = (x - A)^2 + B.$$

Sketch the curve $y = Q(x)$.

- B2** Determine the modulus and principal argument of the complex numbers $z_1 = -3 + 4i$ and $z_2 = -4 - 3i$.
By expressing z_1 and z_2 in exponential form, or otherwise, show that

$$\frac{z_1^3}{z_2^4} = \frac{1}{25} \bar{z}_1,$$

where \bar{z}_1 is the complex conjugate of z_1 .

- B3** Find the values of x at which the function

$$f(x) = x^3 + 2x^2 - 4x + 1$$

has stationary points. Determine the nature of each of the stationary points and sketch the curve $y = f(x)$.

- B4** A function $f(x, y)$ of two variables is given by

$$f(x, y) = \left(2x^5 - \frac{1}{x^5}\right) \cos 5y.$$

Calculate the first-order partial derivatives f_x and f_y . Calculate the second-order partial derivatives f_{xx} and f_{yy} and demonstrate that

$$x^2 f_{xx} + x f_x + f_{yy} = 0.$$

- B5** Let

$$I = \int \frac{dx}{x^2 + 2x + a}$$

where a is a constant. Find the indefinite integral I in the following cases,

- (i) $a = 1$,
- (ii) $a = 0$,
- (iii) $a = 2$.

B6 Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & -4 & 1 \end{bmatrix}.$$

- (a) Find $|A|$, the determinant of A .
- (b) Find A^{-1} , the inverse of A .
- (c) Use A^{-1} to find x , y and z which are solutions to the linear equations

$$\begin{aligned} x + 2z &= 5, \\ 3y + z &= 11, \\ x - 4y + z &= -9. \end{aligned}$$

B7 Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

- (a) Find the eigenvalues and normalized eigenvectors of A .
- (b) Show that the eigenvectors are orthogonal.
- (c) Four points $PQRS$ on a unit square are described by the column vectors

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

respectively. The transformed points $P'Q'R'S'$ are obtained by multiplying the column vectors by matrix A . Draw a sketch of the xy plane showing the original points $PQRS$, the transformed points $P'Q'R'S'$, and the direction of the eigenvectors of A .

B8 Consider the following differential equation:

$$y'' + 2y' + 2y = f(x)$$

- (a) Find the general solution in the case $f(x) = 0$.
- (b) Find a particular solution for $f(x) = \cos x$.
- (c) Find the solution for $f(x) = \cos x$ with initial conditions $y(0) = y'(0) = 0$.

End of Question Paper

MAS140-151-152 Formula Sheet

These results may be quoted without proof unless proofs are asked for in the questions.

Trigonometry

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x$$

$$2 \sin^2 x = 1 - \cos 2x$$

$$2 \cos^2 x = 1 + \cos 2x$$

$$2 \sin x \cos x = \sin 2x$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$a \cos x + b \sin x = R \cos(x - \alpha)$$

$$\text{where } R = \sqrt{a^2 + b^2} \quad ,$$

$$\cos \alpha = \frac{a}{R} \quad \text{and} \quad \sin \alpha = \frac{b}{R}$$

$$2 \cos x \cos y = \cos(x + y) + \cos(x - y)$$

$$2 \sin x \sin y = \cos(x - y) - \cos(x + y)$$

$$2 \sin x \cos y = \sin(x + y) + \sin(x - y)$$

$$\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$$

$$\sin x = \frac{2 \tan(x/2)}{1 + \tan^2(x/2)}$$

$$\tan x = \frac{2 \tan(x/2)}{1 - \tan^2(x/2)}$$

Hyperbolic Functions

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{coth} x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$2 \cosh^2 x = 1 + \cosh 2x$$

$$2 \sinh^2 x = \cosh 2x - 1$$

$$2 \sinh x \cosh x = \sinh 2x$$

$$\operatorname{sech}^2 x = 1 - \tanh^2 x$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}), \text{ all } x$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \quad x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right), \quad |x| < 1$$

Series

Sum of an arithmetic series:

$$\frac{\text{first term} + \text{last term}}{2} \times (\text{number of terms})$$

Sum of a geometric series: $1 + x + x^2 + \dots + x^{n-1} = \frac{1 - x^n}{1 - x}$

Binomial theorem: $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \binom{n}{r}x^r + \dots$

$$\text{where } \binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

If n is a positive integer then the series terminates and the result is true for all x , otherwise, the series is infinite and only converges for $|x| < 1$.

$$\left. \begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ \sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \\ \cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \\ \exp x &= e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \end{aligned} \right\} \text{valid for all } x$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (-1 < x \leq 1)$$

Differentiation

<u>Function</u>	<u>Derivative</u>	<u>Function</u>	<u>Derivative</u>
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\operatorname{cosec}^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}, x < 1$	$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}, x < 1$
$\tan^{-1} x$	$\frac{1}{1+x^2}$	$\cot^{-1} x$	$-\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$
$\tanh x$	$\frac{1}{\cosh^2 x}$	$\operatorname{coth} x$	$-\frac{1}{\sinh^2 x}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$	$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}, x > 1$
$\tanh^{-1} x$	$\frac{1}{1-x^2}, x < 1$		
$\operatorname{coth}^{-1} x$	$\frac{1}{1-x^2}, x > 1$		

Integration

In the following table the constants of integration have been omitted.

$$\int x^n dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$$

$$\int \frac{dx}{x} = \ln |x|$$

$$\int e^x dx = e^x$$

$$\int a^x dx = \frac{a^x}{\ln a} \quad (a > 0, a \neq 1)$$

$$\int \sin x dx = -\cos x$$

$$\int \cos x dx = \sin x$$

$$\int \sec^2 x dx = \tan x$$

$$\int \operatorname{cosec}^2 x dx = -\cot x$$

$$\int \sinh x dx = \cosh x$$

$$\int \cosh x dx = \sinh x$$

$$\int \operatorname{sech}^2 x dx = \tanh x$$

$$\int \operatorname{cosech}^2 x dx = -\coth x$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} \quad (|x| < a)$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} \quad (|x| > a)$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| \quad (= \tanh^{-1} \frac{x}{a} \quad \text{if } |x| < a)$$

$$\int \operatorname{cosec} x dx = \ln \tan \left(\frac{x}{2} \right) \quad \text{or} \quad \ln (\operatorname{cosec} x - \cot x)$$

$$\int \sec x dx = \ln \tan \left(\frac{x}{2} + \frac{\pi}{4} \right) \quad \text{or} \quad \ln (\sec x + \tan x)$$

$$\int \operatorname{cosech} x dx = \ln \tanh \left(\frac{x}{2} \right)$$

Integration by parts

$$\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx$$

Newton-Leibnitz formula

If $F'(x) = f(x)$, then $\int_a^b f(x) dx = F(b) - F(a) = F(x) \Big|_a^b$

Variable substitution in definite integral

If $x = \varphi(t)$ is a monotonic function in the interval $[\alpha, \beta]$ and $a = \varphi(\alpha)$, $b = \varphi(\beta)$, then

$$\int_a^b f(x) dx = \int_\alpha^\beta f(\varphi(t)) \varphi'(t) dt$$

Variable substitution for a rational function of sin x and cos x

Let $t = \tan\left(\frac{x}{2}\right)$ then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $\frac{dx}{dt} = \frac{2}{1+t^2}$.

Area of planar figure

The area of a planar figure bounded by the graph of a continuous positive function $f(x)$, the x -axis, and the ordinates $x = a$ and $x = b$ is

$$S = \int_a^b f(x) dx$$

Volume of solid of revolution

The volume of a solid of revolution, obtained by rotation of a planar figure bounded by the graph of a continuous positive function $f(x)$, the x -axis, and the ordinates $x = a$ and $x = b$ through a complete revolution around the x -axis, is

$$V = \pi \int_a^b f^2(x) dx$$