The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012–2013**

MAS152 Essential Mathematical Skills and Techniques **3 hours**

*Attempt **ALL** questions.*

Each question in Section A carries 3 marks, each question in Section B carries 8 marks.

Section A

A1 Use the binomial theorem to find the value of a for which $\lim_{x\to 0} \dfrac{(8-ax)^{1/3}-2}{x} = 1$.

A2 If $f(x) = \dfrac{3x}{x+2}$, $x \geq 0$, what is the range of $f(x)$? Find $f^{-1}(y)$.

A3 Find the first three nonzero terms in the Maclaurin expansion of $f(x) = e^{-2x}\cos 3x$.

A4 Evaluate $\lim_{x\to\pi/2} \dfrac{x\cos 3x}{2x-\pi}$.

A5 Find all the complex numbers z for which $\mathrm{Re}(z) + \mathrm{Im}(z) = 0$ and $|z-1+i| = \sqrt{2}$.

A6 If $\mathbf{a} = (3, 1, -1)$ and $\mathbf{b} = (1, -2, 1)$, show that $\mathbf{a}$ is perpendicular to $\mathbf{b}$ and find $\mathbf{a} \times \mathbf{b}$.

A7 Use integration by parts to find the indefinite integral

$$\int x^n \ln(x) dx$$

where $n \neq -1$.

A8 Compute the definite integral

$$\int_{-1}^{1} \sin^2 \left(\frac{\pi x}{2} \right) dx.$$

A9 Find the determinant of $A^{-1}B$, where A and B are given by

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -4 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}.$$

A10 Consider the homogeneous linear system of equations

$$\begin{aligned} 3x + y &= 0, \\ ax - 4y &= 0, \end{aligned}$$

which has the trivial solution $x = y = 0$. Find the value of a for which the system has additional solutions.

A11 Find the general solution of the following differential equation:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0.$$

A12 Show that the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{1 - x^2},$$

is given by $y = A\sqrt{\dfrac{1 + x}{1 - x}}$, where A is a constant.

Section B

B1 Find the quadratic $Q(x)$ for which the polynomial $P(x) = x^3 - x^2 - x - 15$ can be written in the form

$$P(x) = (x-3)\,Q(x).$$

Find the roots of $Q(x)$ and plot them on an Argand diagram. Find the constants A and B such that

$$Q(x) = (x-A)^2 + B.$$

Sketch the curve $y = Q(x)$.

B2 Determine the modulus and principal argument of the complex numbers $z_1 = -3 + 4i$ and $z_2 = -4 - 3i$.
By expressing z_1 and z_2 in exponential form, or otherwise, show that

$$\frac{z_1^3}{z_2^4} = \frac{1}{25}\bar{z_1},$$

where $\bar{z_1}$ is the complex conjugate of z_1.

B3 Find the values of x at which the function

$$f(x) = x^3 + 2x^2 - 4x + 1$$

has stationary points. Determine the nature of each of the stationary points and sketch the curve $y = f(x)$.

B4 A function $f(x, y)$ of two variables is given by

$$f(x,y) = \left(2x^5 - \frac{1}{x^5}\right)\cos 5y.$$

Calculate the first-order partial derivatives f_x and f_y. Calculate the second-order partial derivatives f_{xx} and f_{yy} and demonstrate that

$$x^2 f_{xx} + x f_x + f_{yy} = 0.$$

B5 Let

$$I = \int \frac{dx}{x^2 + 2x + a}$$

where a is a constant. Find the indefinite integral I in the following cases,

(i) $a = 1$,

(ii) $a = 0$,

(iii) $a = 2$.

B6 Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & -4 & 1 \end{bmatrix}.$$

(a) Find $|A|$, the determinant of A.

(b) Find A^{-1}, the inverse of A.

(c) Use A^{-1} to find x, y and z which are solutions to the linear equations

$$\begin{aligned} x + 2z &= 5, \\ 3y + z &= 11, \\ x - 4y + z &= -9. \end{aligned}$$

B7 Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues and normalized eigenvectors of A.

(b) Show that the eigenvectors are orthogonal.

(c) Four points $PQRS$ on a unit square are described by the column vectors

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

respectively. The transformed points $P'Q'R'S'$ are obtained by multiplying the column vectors by matrix A. Draw a sketch of the xy plane showing the original points $PQRS$, the transformed points $P'Q'R'S'$, and the direction of the eigenvectors of A.

B8 Consider the following differential equation:

$$y'' + 2y' + 2y = f(x)$$

(a) Find the general solution in the case $f(x) = 0$.

(b) Find a particular solution for $f(x) = \cos x$.

(c) Find the solution for $f(x) = \cos x$ with initial conditions $y(0) = y'(0) = 0$.

End of Question Paper

MAS140-151-152 Formula Sheet

These results may be quoted without proof unless proofs are asked for in the questions.

Trigonometry

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \text{cosec}^2 x$

$2\sin^2 x = 1 - \cos 2x$

$2\cos^2 x = 1 + \cos 2x$

$2\sin x \cos x = \sin 2x$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

$a\cos x + b\sin x = R\cos(x - \alpha)$

where $R = \sqrt{a^2 + b^2}$,

$\cos\alpha = \frac{a}{R}$ and $\sin\alpha = \frac{b}{R}$

$2\cos x \cos y = \cos(x+y) + \cos(x-y)$

$2\sin x \sin y = \cos(x-y) - \cos(x+y)$

$2\sin x \cos y = \sin(x+y) + \sin(x-y)$

$\cos x = \dfrac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$

$\sin x = \dfrac{2\tan(x/2)}{1 + \tan^2(x/2)}$

$\tan x = \dfrac{2\tan(x/2)}{1 - \tan^2(x/2)}$

Hyperbolic Functions

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\tanh x = \dfrac{\sinh x}{\cosh x}$

$\coth x = \dfrac{\cosh x}{\sinh x}$

$\text{sech} x = \dfrac{1}{\cosh x}$

$\cosh^2 x - \sinh^2 x = 1$

$2\cosh^2 x = 1 + \cosh 2x$

$2\sinh^2 x = \cosh 2x - 1$

$2\sinh x \cosh x = \sinh 2x$

$\text{sech}^2 x = 1 - \tanh^2 x$

$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$, all x

$\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$, $x \geq 1$

$\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$, $|x| < 1$

Series

Sum of an arithmetic series:

$$\frac{\textit{first term} \ + \ \textit{last term}}{2} \times (\textit{number of terms})$$

Sum of a geometric series: $1 + x + x^2 + \ldots + x^{n-1} = \dfrac{1 - x^n}{1 - x}$

Binomial theorem: $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots + \dbinom{n}{r}x^r + \ldots$

$$\text{where } \binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

If n is a positive integer then the series terminates and the result is true for all x, otherwise, the series is infinite and only converges for $|x| < 1$.

$$\left.\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \\
\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \\
\exp x &= e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
\end{aligned}\right\} \text{valid for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad (-1 < x \le 1)$$

Differentiation

Function	Derivative	Function	Derivative
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\text{cosec}^2 x$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}\,,\ \lvert x\rvert < 1$	$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}\,,\ \lvert x\rvert < 1$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$	$\cot^{-1} x$	$-\dfrac{1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$
$\tanh x$	$\dfrac{1}{\cosh^2 x}$	$\coth x$	$-\dfrac{1}{\sinh^2 x}$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{x^2+1}}$	$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}\,,\ \lvert x\rvert > 1$
$\tanh^{-1} x$	$\dfrac{1}{1-x^2}\,,\ \lvert x\rvert < 1$		
$\coth^{-1} x$	$\dfrac{1}{1-x^2}\,,\ \lvert x\rvert > 1$		

Integration

In the following table the constants of integration have been omitted.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$$

$$\int \frac{dx}{x} = \ln|x|$$

$$\int e^x\,dx = e^x$$

$$\int a^x\,dx = \frac{a^x}{\ln a} \quad (a > 0,\ a \neq 1)$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int \sec^2 x\,dx = \tan x$$

$$\int \mathrm{cosec}^2 x\,dx = -\cot x$$

$$\int \sinh x\,dx = \cosh x$$

$$\int \cosh x\,dx = \sinh x$$

$$\int \mathrm{sech}^2 x\,dx = \tanh x$$

$$\int \mathrm{cosech}^2 x\,dx = -\coth x$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} \quad (|x| < a)$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} \quad (|x| > a)$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| \quad \left(= \tanh^{-1}\frac{x}{a} \quad \text{if } \ |x| < a\right)$$

$$\int \mathrm{cosec} x\,dx = \ln\tan\left(\frac{x}{2}\right) \quad \text{or} \quad \ln\left(\mathrm{cosec} x - \cot x\right)$$

$$\int \sec x\,dx = \ln\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) \quad \text{or} \quad \ln\left(\sec x + \tan x\right)$$

$$\int \mathrm{cosech} x\,dx = \ln\tanh\left(\frac{x}{2}\right)$$

Integration by parts

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx$$

Newton-Leibnitz formula

If $F'(x) = f(x)$, then $\displaystyle\int_a^b f(x)\, dx = F(b) - F(a) = F(x)\Big|_a^b$

Variable substitution in definite integral

If $x = \varphi(t)$ is a monotonic function in the interval $[\alpha, \beta]$ and $a = \varphi(\alpha)$, $b = \varphi(\beta)$, then

$$\int_a^b f(x)\, dx = \int_\alpha^\beta f(\varphi(t))\, \varphi'(t)\, dt$$

Variable substitution for a rational function of sin x and cos x

Let $t = \tan\left(\frac{x}{2}\right)$ then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $\frac{dx}{dt} = \frac{2}{1+t^2}$.

Area of planar figure

The area of a planar figure bounded by the graph of a continuous positive function $f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$ is

$$S = \int_a^b f(x)\, dx$$

Volume of solid of revolution

The volume of a solid of revolution, obtained by rotation of a planar figure bounded by the graph of a continuous positive function $f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$ through a complete revolution around the x-axis, is

$$V = \pi \int_a^b f^2(x)\, dx$$