



The
University
Of
Sheffield.

MAS153/MAS159

SCHOOL OF MATHEMATICS AND STATISTICS

**Spring Semester
2012–2013**

**Mathematics (Materials)
Mathematics For Chemists**

3 hours

All questions are compulsory. The marks awarded to each question or section of question are shown in italics.

- 1 Find the constants r and α , with $r > 0$ and $-\pi < \alpha \leq \pi$, such that

$$\cos x + \sqrt{3} \sin x = r \cos(x - \alpha). \quad (4 \text{ marks})$$

Hence find all solutions x in $(-\pi, \pi]$ of

$$\cos x + \sqrt{3} \sin x = \sqrt{2}. \quad (4 \text{ marks})$$

- 2 Solve the following for x :

$$2 \log_a x - \log_a(x-1) = \log_a(x-2). \quad (6 \text{ marks})$$

- 3 Identify the number d such that

$$\frac{3^9 4^{9/2}}{6(3^4)^2} = 2^d \quad (4 \text{ marks})$$

- 4 Simplify

$$(5x^2 y^{-\frac{3}{2}} z^{\frac{1}{4}})^2 \times (4x^4 y^2 z)^{-1/2} \quad (4 \text{ marks})$$

- 5 Factorise $3x^2 + 14x + 8$. (2 marks)

6 Verify the following identity

$$1 - \frac{\sin \theta \tan \theta}{1 + \sec \theta} = \cos \theta \quad (5 \text{ marks})$$

7 State the range of the function $f(x) = x^2 - 1$. (2 marks)

8 State the domain of the function

$$y(x) = \begin{cases} x^2 + 1 & -1 \leq x \leq 2 \\ 3x & 2 < x \leq 6 \\ 2x + 1 & x > 6. \end{cases} \quad (2 \text{ marks})$$

9 For the circle $x^2 + y^2 = 5$, find the equation of the tangent which meets the circle at $(-2, 1)$. (6 marks)

Find where this tangent intersects the circle with centre $(-3, 9)$ and radius 5. (7 marks)

10 If $y = a \cos(\omega x) + b \sin(\omega x)$, find $\frac{d^2y}{dx^2}$ in terms of y . (4 marks)

11 (i) Showing your working clearly, find the coefficient of x^3 in the expansion of $(1 + x)^{14}$. (2 marks)

(ii) Use the binomial theorem to evaluate

$$\lim_{x \rightarrow 0} \frac{x^2}{\sqrt{4 - x^2} - 2}. \quad (3 \text{ marks})$$

12 (i) Find the angle between the vectors $\mathbf{a} = (1, 0, 3)$ and $\mathbf{b} = (2, 1, -1)$, giving your answer in radians to two decimal places. (3 marks)

(ii) A plane passes through the points $(1, 2, 1)$, $(0, 1, 0)$ and $(-1, 3, 2)$. Find the Cartesian equation of the plane. (6 marks)

13 Prove, from the definitions of $\sinh x$ and $\cosh x$, the identity

$$\sinh^2 x + \cosh^2 x = \cosh 2x. \quad (3 \text{ marks})$$

- 14** (i) If $y = \ln(\sinh x)$, show that

$$\frac{dy}{dx} = \coth x. \quad (2 \text{ marks})$$

- (ii) If $y = \frac{\sinh 2x}{\cosh x}$, show that

$$\frac{dy}{dx} = 2 \cosh x. \quad (4 \text{ marks})$$

- 15** By making the substitution $x = a \sinh \theta$, where a is a positive constant, show that

$$\int_0^a \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} 1. \quad (6 \text{ marks})$$

- 16** Find the Maclaurin series for $\ln(1 - x)$, as far as the term in x^4 . (7 marks)

- 17** Complex numbers z_1 and z_2 are defined by

$$z_1 = 1 + i, \quad z_2 = 3 - i.$$

Find, in the form $a + bi$ where a and b are real,

(i) z_1^4 (2 marks)

(ii) $\frac{z_2}{z_1 + 2z_2}$ (3 marks)

- 18** A set of linear equations can be written as

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ 8 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 2 & -1 & -2 \\ 3 & 1 & 2 \\ 1 & 2 & -4 \end{pmatrix}.$$

Find the inverse, A^{-1} , of A , and use it to find the values of x , y and z which satisfy the equations. (9 marks)

End of Question Paper

Formula Sheet for MAS153/MAS159 Examination

These results may be quoted without proof, unless proofs are asked for in the question.

Trigonometry

For any angles A and B

$$\sin^2 A + \cos^2 A = 1$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$$

Coordinate Geometry

The acute angle α between lines with gradients m_1 and m_2 satisfies

$$\tan \alpha = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| \quad (m_1 m_2 \neq -1)$$

while the lines are perpendicular if $m_1 m_2 = -1$.

The equation of a circle centre (x_0, y_0) and radius a is $(x - x_0)^2 + (y - y_0)^2 = a^2$.

Hyperbolic Functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\operatorname{sech}^2 x + \tanh^2 x = 1$$

$$\cosh^2 x + \sinh^2 x = \cosh 2x$$

$$2 \sinh x \cosh x = \sinh 2x$$

$$\cosh^2 x = (1 + \cosh 2x)/2$$

$$\sinh^2 x = -(1 - \cosh 2x)/2$$

Differentiation

<u>Function</u> (y)	<u>Derivative</u> (dy/dx)
x^n	nx^{n-1}
$\sin ax$	$a \cos ax$
$\cos ax$	$-a \sin ax$
$\tan ax$	$a \sec^2 ax$
e^{ax}	ae^{ax}
$\ln(ax)$	$\frac{1}{x}$
$\ln f(x)$	$\frac{f'(x)}{f(x)}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

NB. It is assumed that x takes only those values for which the functions are defined.

For u and v functions of x , and with $u' = \frac{du}{dx}$, $v' = \frac{dv}{dx}$,

$$\frac{d}{dx}(uv) = uv' + vu',$$

while

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{vu' - uv'}{v^2}.$$

For $y = y(t)$, $t = t(x)$,

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx}.$$

Integration

In the following table the constants of integration have been omitted.

<u>Function</u> $f(x)$	<u>Integral</u> $\int f(x) dx$
x^n	$\frac{x^{n+1}}{n+1} \quad n \neq -1$
ae^{ax}	e^{ax}
$\frac{1}{x}$	$\ln x $
$a \sin ax$	$-\cos ax$
$a \cos ax$	$\sin ax$
$a \tan ax$	$\ln \sec ax $
$\frac{1}{a^2 + x^2}$	$\frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$
$\frac{1}{a^2 - x^2}$	$\frac{1}{a} \tanh^{-1} \left(\frac{x}{a} \right)$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1} \left(\frac{x}{a} \right)$
$\frac{1}{\sqrt{x^2 + a^2}}$	$\sinh^{-1} \left(\frac{x}{a} \right)$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\cosh^{-1} \left(\frac{x}{a} \right)$
$\frac{f'(x)}{f(x)}$	$\ln f(x) $

Integration by parts

$$\int uV dx = (\text{integral of } V) \times u - \int (\text{integral of } V) \times \frac{du}{dx} dx$$

or

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx.$$

Series

Binomial Theorem: $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \binom{n}{r}x^r + \dots$

$$\text{where } \binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

If n is a positive integer, the series terminates and is convergent for all x .

If n is not a positive integer, the series is infinite and converges for $|x| < 1$.

Taylor expansion of $f(x)$ about $x = a$ is

$$f(a) + (x-a)f^{(1)}(a) + \frac{(x-a)^2}{2!}f^{(2)}(a) + \dots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \dots$$

Maclaurin expansion of $f(x)$ is

$$f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots$$

Alternating Series Test

The series $a_1 - a_2 + a_3 - a_4 + \dots$, where $a_1, a_2, a_3, a_4, \dots$ are all positive, converges if $a_1 > a_2 > a_3 > \dots$ and $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Ratio Test

If the series $a_1 + a_2 + a_3 + a_4 + \dots$ satisfies

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lambda,$$

then

1. if $\lambda > 1$, the series diverges,
2. if $\lambda < 1$, the series converges.

Vectors

If vectors \mathbf{a} and \mathbf{b} are given in Cartesian component form by $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, then

the scalar product $\mathbf{a} \cdot \mathbf{b}$ is given by

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

and the vector product $\mathbf{a} \times \mathbf{b}$ is given by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

If a plane passes through a point with position vector \mathbf{a} , and is normal to the vector \mathbf{n} , then the equation of the plane is

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n},$$

where $\mathbf{r} = (x, y, z)$.