Data provided: Formula sheet

MAS165

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012-2013**

Mathematics for Physicists **2 hours**

You should attempt ALL questions of this exam.

Section A

A1 A plane is given by the equation

$$4x + 5y + 7z = 21$$

and a line by the equation $\mathbf{r} = (1, 2, 3) + \lambda(1, 2, -2)$, where λ is a real parameter.

(i) Show that the line does not intersect the plane. ***(4 marks)***

(ii) Therefore, calculate the distance of the line to the plane. ***(4 marks)***

(iii) Find the direction of the line of intersection of the two planes $x+3y-z = 5$ and $2(x - y) + 4z = 3$. ***(5 marks)***

A2 Show that $f(x, y) = e^{-x}\cos(y) - e^{-y}\cos(x)$ obeys the two dimensional Laplace equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0\ .$$

Show that the function $g(x, y) = \sqrt{x^2 + y} - xy$ does not obey the two dimensional Laplace equation. ***(9 marks)***

A3 Stokes' theorem may be written:

$$\oint_C \mathbf{G}.\mathbf{dr} = \int_S (\nabla \times \mathbf{G}).\hat{\mathbf{n}}\, dS$$

Indicate whether the following statements about Stokes' theorem, as expressed here, are true or false

(i) The term $(\nabla \times \mathbf{G})$ is the divergence of the vector field $\mathbf{G}$.

(ii) $\hat{\mathbf{n}}$ is a unit vector parallel with the boundary C.

(iii) $\int_S dS$ is a surface integral, over the surface S.

(3 marks)

Section B

B1 (i) Find the work done by a force $\mathbf{F} = (x + yz)\mathbf{i} + (y + xz)\mathbf{j} + (z + xy)\mathbf{k}$ in moving a particle from the origin $\mathcal{O}$ to the point $A(1, 1, 1)$

(a) along the curve $x = t, y = t^2, z = t^3$,

(b) along the straight line $\mathcal{O}A$. ***(12 marks)***

(ii) A scalar function is given as

$$\phi(x, y, z) = x^2 - y \sin(x - z).$$

(a) Calculate the gradient of $\phi(x, y, z)$, i.e. calculate $\mathbf{V} = \nabla\phi$. ***(3 marks)***

(b) Using your result, calculate the divergence of $\mathbf{V}$. ***(4 marks)***

(c) By explicit calculation, show that $\nabla \times \mathbf{V} = 0$. ***(6 marks)***

B2 (i) A vector field is given by

$$\mathbf{V} = V_1\hat{\mathbf{r}} + V_2\hat{\boldsymbol{\theta}} + V_3\hat{\mathbf{z}} = r\hat{\mathbf{r}} + (a + r^3)\hat{\boldsymbol{\theta}} + b\ln(z)\hat{\mathbf{z}}$$

in cylindrical polar coordinates, where a and b are positive constants. Calculate the divergence and curl of the vector field, given that the divergence and curl may be expressed in cylindrical coordinates as

$$\nabla.\mathbf{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_1) + \frac{1}{r}\frac{\partial}{\partial \theta}(V_2) + \frac{\partial}{\partial z}(V_3)$$

and

$$\nabla \times \mathbf{V} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ V_1 & rV_2 & V_3 \end{vmatrix}$$

respectively. Can $\nabla \times \mathbf{V}$ be zero? ***(10 marks)***

(ii) Sketch the region of integration represented by the repeated integral

$$\int\int_R x^2 y \, dx \, dy$$

where R is the region such that $x \geq 0$, $y \geq 0$, and $x^2 + y^2 \leq a^2$. By transforming to plane polar coordinates, evaluate the integral.

(15 marks)

B3 (i) Verify the divergence theorem

$$\int_V (\nabla \cdot \mathbf{A}) \, dV = \oint_S \mathbf{A} \cdot \hat{\mathbf{n}} \, dS \ ,$$

for the vector field $\mathbf{A} = (x, y, z)$ and S being the surface enclosing a cylinder of radius a (i.e. $x^2 + y^2 = a^2$) and height h. The bottom surface of the cylinder lies in the $xy-$plane ($z = 0$). Hint: split the surface integral into three parts. To find $\hat{\mathbf{n}}$ for the curved surface of the cylinder, note that you can find it by calculating $\nabla\phi$, with $\phi(x, y) = x^2 + y^2 = a^2$ describing the surface. ***(15 marks)***

(ii) A magnetic field is given, in cylindrical polar coordinates (r, θ, z), as $\mathbf{H} = H_0 r^2 \hat{\boldsymbol{\theta}}/a^2$, with $r \leq a$, where H_0 and a are positive constants. The magnetic field vanishes for $r > a$. Evaluate

$$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{x},$$

where $\mathcal{C}$ is the circle $z = 0$, $r = R$, described in the anticlockwise sense for $R < a$.

(10 marks)

End of Question Paper

Mathematical Formulae:

Spherical Polar Coordinates:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$

$$dV = r^2\sin\theta dr d\theta d\phi \quad \text{(Element of volume)}$$

In the following $\mathbf{F} = F_1\hat{\mathbf{r}} + F_2\hat{\boldsymbol{\theta}} + F_3\hat{\boldsymbol{\phi}}$ (note that $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ are unit vectors):

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2F_1\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_2) + \frac{1}{r\sin\theta}\frac{\partial F_3}{\partial\phi}$$

and

$$\nabla\times\mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_1 & rF_2 & r\sin\theta F_3 \end{vmatrix}.$$

Let f be a scalar function, then the gradient is given by

$$\text{grad} f = \nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}}.$$

Plane Polar Coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad dA = dxdy = rdrd\theta$$

Vector Calculus:

$$\begin{aligned}
\nabla\phi &= \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} \\
\nabla\cdot\mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
\nabla\times\mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k} \\
\nabla^2\phi &= \nabla\cdot(\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \\
\nabla\times(\nabla\phi) &= 0 \\
\nabla\cdot(\nabla\times\mathbf{A}) &= 0 \\
\nabla\times(\nabla\times\mathbf{A}) &= \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}
\end{aligned}$$

Vectors:

$$
\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= A_xB_x + A_yB_y + A_zB_z \\
\mathbf{A}\times\mathbf{B} &= (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k} \\
\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \\
\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})
\end{aligned}
$$

Trigonometry:

$$
\begin{aligned}
\sin(\phi\pm\theta) &= \sin\phi\cos\theta \pm \cos\phi\sin\theta \\
\cos(\phi\pm\theta) &= \cos\phi\cos\theta \mp \sin\phi\sin\theta \\
\tan(\theta\pm\phi) &= \frac{\tan\phi\pm\tan\theta}{1\mp\tan\phi\tan\theta} \\
\sin(2\phi) &= 2\sin\phi\cos\phi \\
\cos(2\phi) &= 2\cos^2\phi - 1 = 1 - 2\sin^2\phi \\
\sin\phi + \sin\theta &= 2\sin\left(\frac{\phi+\theta}{2}\right)\cos\left(\frac{\phi-\theta}{2}\right) \\
\sin\phi - \sin\theta &= 2\cos\left(\frac{\phi+\theta}{2}\right)\sin\left(\frac{\phi-\theta}{2}\right) \\
\cos\phi + \cos\theta &= 2\cos\left(\frac{\phi+\theta}{2}\right)\cos\left(\frac{\phi-\theta}{2}\right) \\
\cos\phi - \cos\theta &= 2\sin\left(\frac{\phi+\theta}{2}\right)\sin\left(\frac{\phi-\theta}{2}\right)
\end{aligned}
$$