The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2012–13

MECHANICS — 2 hours

Attempt all THREE questions.

1 (i) Evaluate the double integral $I = \int\int_R x^2 + y^2 \; dx \; dy$ over the triangular region R with vertices $(0,0)$, $(1,1)$ and $(2,0)$. You are advised to sketch the region over which the integration takes place. ***(13 marks)***

(ii) By transforming to polar co-ordinates, evaluate the double integral of

$$I = \int\int_D \exp[-(x^2+y^2)] \; dx \; dy$$

where D is the region between the two circles $x^2+y^2=1$ and $x^2+y^2=4$. You may leave your answer in terms of π and e. ***(12 marks)***

2 (i) A semi-circular lamina of radius a has uniform density, σ. The $y-$axis lies along the line of symmetry of the lamina and the $x-$axis lies along its straight edge. By dividing the lamina into thin strips of thickness δy parallel to the $x-$axis, derive the result that the position of its centre of mass has co-ordinates $(\overline{x}, \overline{y}) = (0, \frac{4a}{3\pi})$. ***(12 marks)***

(ii) Find, in regular cartesian form, the equation of the tangent plane to the surface $x^2+y^2+z^2=30$ at the point $Q=(-2,1,5)$. ***(6 marks)***

(iii) A light string passes over a frictionless pulley whose axis is fixed. A mass m is attached to one end of the string and a mass $2m$ is attached to the other end. Both masses hang freely. Given that the two masses are level at the moment of release from rest, find the velocity of each when the mass $2m$ is 1 m lower then the mass m. You may take the acceleration due to gravity to be 9.81 ms^{-2}. ***(7 marks)***

3 (i) With reference to plane polar co-ordinates, you are given that the acceleration of a planet around its star can be expressed as

$$\ddot{\mathbf{r}} = -\mu \mathbf{r} r^{-3},$$

where μ is a constant. The vector $\mathbf{h}$, which is taken to be constant, is defined by

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}.$$

You are reminded that

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}.$$

(a) Show that

$$\mathbf{h} \times \ddot{\mathbf{r}} = -\mu \frac{d}{dt}(\mathbf{r} r^{-1}).$$

(4 marks)

(b) Hence deduce that

$$\mathbf{h} \times \dot{\mathbf{r}} = -\mu \mathbf{r} r^{-1} - \mu \mathbf{e},$$

where $\mathbf{e}$ denotes a constant vector. *(2 marks)*

(ii) A satellite of mass m orbits the Earth, which has mass M and radius a. When the satellite is at a distance $3a$ from the centre of the Earth it is brought suddenly to rest by a collision with a meteorite. Subsequently it falls directly to Earth. Ignoring air resistance, find the speed of the satellite in ms^{-1} when it reaches the Earth's surface. You are given that $a = 6.378 \times 10^6$ m and $GM = 3.986 \times 10^{14}$ $\mathrm{m^3 s^{-2}}$, where G is the gravitational constant. *(5 marks)*

(iii) A compound pendulum consists of a uniform thin rod of mass m and length $2a$, to which is clamped a uniform circular disc of mass $6m$ and radius $\frac{a}{3}$. The disc is in a vertical plane with its centre on the rod a distance x from the end, O, about which the pendulum swings in the vertical plane. You are given that the moment of inertia of a rod of mass M and length L about a perpendicular axis passing through its mid-point is $\frac{ML^2}{12}$ and that the moment of inertia of a disc of mass M and radius r about an axis perpendicular to its plane and passing through its centre is $\frac{Mr^2}{2}$.

(a) Draw a sketch of the pendulum. Show that the moment of inertia about the point O of the compound pendulum is $m\left(\frac{5}{3}a^2 + 6x^2\right)$. *(7 marks)*

(b) Find the period of small oscillations of the pendulum. *(7 marks)*

End of Question Paper