The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2012-2013

NUMERICAL LINEAR ALGEBRA — 2 hours

Answer FOUR questions. You are advised not to answer more than four questions: if you do, only your best four will be counted.

Please leave this exam paper on your desk
Do not remove it from the hall

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1 (i) Given a set of data (x_j, f_j), $j = 0, 1, \ldots, m$ and the basis functions $\phi_0(x)$, $\phi_1(x)$, ..., $\phi_n(x)$, we use

$$Y(x) = \sum_{i=0}^{n} a_i \phi_i(x)$$

to approximate the data in the least-squares sense, where a_i, $i = 0, 1, \ldots, n$ are the coefficients. Show that the normal equations for the coefficients are

$$\sum_{i=0}^{n} \left[a_i \sum_{j=0}^{m} \phi_i(x_j)\phi_k(x_j) \right] = \sum_{j=0}^{m} f_j \phi_k(x_j),$$

for $k = 0, 1, 2, \ldots, n$.

(10 marks)

(ii) Using a suitable transformation, determine a least-squares fit of the form $Y(x) = A + \ln(B + x)$ to the data:

j :	0	1	2	3
x_j :	1.5	2.5	3.5	4.5
f_j :	2.9263	3.2328	3.7041	3.2007

where A and B are the coefficients to be determined.

(15 marks)

2 Let A and B be square matrices, and $\mathbf{x}$ a vector. $||\mathbf{x}||$ denotes the norm of $\mathbf{x}$, and $||A||$ and $||B||$ the subordinate norms of A and B, respectively. The following formulae are given:

$$||A\mathbf{x}|| \leq ||A||\,||\mathbf{x}||, \quad \text{and} \quad ||AB|| \leq ||A||\,||B||.$$

(i) Write down the definition of the condition number $\mathcal{K}(A)$ of a matrix A, and show that $\mathcal{K}(A) \geq 1$. ***(6 marks)***

(ii) Consider a linear system $A\mathbf{x} = \mathbf{b}$, for which $\mathbf{x}^e$ is the exact solution. Suppose that $\mathbf{b}$ is perturbed by some small error $\delta\mathbf{b}$, so that practically we are solving $A\mathbf{x} = \mathbf{b} + \delta\mathbf{b}$. Let $\hat{\mathbf{x}}$ be an approximate solution of the perturbed system, and $\mathbf{r} = A\hat{\mathbf{x}} - \mathbf{b} - \delta\mathbf{b}$ be the residual error. Show that the relative error in $\hat{\mathbf{x}}$ has the following bound:

$$\frac{||\hat{\mathbf{x}} - \mathbf{x}^e||}{||\mathbf{x}^e||} \leq \mathcal{K}(A) \frac{||\mathbf{r}|| + ||\delta\mathbf{b}||}{||\mathbf{b}||}.$$

(19 marks)

3 A real square matrix U is an orthogonal matrix if and only if $UU^T = U^T U = I$, where U^T is the transpose of U and I is the identity matrix. Let $\mathbf{w}$, $\mathbf{x}$ and $\mathbf{e}$ be m dimensional column vectors. $\mathbf{w}$ and $\mathbf{x}$ are arbitrary, and $\mathbf{e} = (1, 0, 0, ..., 0)^T$. Recall that the 2-norm of $\mathbf{x}$ is defined as $||\mathbf{x}||_2 = (\mathbf{x}^T\mathbf{x})^{1/2}$. The Householder reflection matrix is defined as

$$P = \left(I - 2\frac{\mathbf{w}\mathbf{w}^T}{\mathbf{w}^T\mathbf{w}} \right).$$

(i) Show that $||U\mathbf{x}||_2 = ||\mathbf{x}||_2$. ***(4 marks)***

(ii) Show that, if $U\mathbf{x} = \alpha\mathbf{e}$ for some constant α, then $|\alpha| = ||\mathbf{x}||_2$. ***(3 marks)***

(iii) Show that $\mathbf{x} - P\mathbf{x} = \beta\mathbf{w}$ for some constant β. Find β. ***(4 marks)***

(iv) Show that $\mathbf{x} + P\mathbf{x}$ is orthogonal to $\mathbf{w}$. ***(6 marks)***

(v) Show that P is an orthogonal matrix. ***(8 marks)***

4 (i) The matrix A has eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ satisfying

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq ... \geq |\lambda_n| > 0$$

with linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$. The eigenvectors have been normalized so that the largest element of each one is unity.

(a) Write down the power iteration for finding the dominant eigenvalue of A and its eigenvector. ***(4 marks)***

(b) Prove that the iteration converges to the dominant eigenvalue and its eigenvector. ***(13 marks)***

(ii) Given matrix

$$A = \begin{pmatrix} 6.0 & 3.4 & 3.3 \\ 8.4 & 7.0 & 1.0 \\ 1.3 & 2.0 & 2.3 \end{pmatrix},$$

use the power method to find the estimate for the dominant eigenvalue and its eigenvector. Start with $\mathbf{z}_0 = (0, -0.2, 1.0)^T$, and perform **two** iterations. State clearly the approximate values for the eigenvalue at each iteration. Work correct to four decimal places. ***(8 marks)***

5 For the equation $A\mathbf{x} = \mathbf{b}$, let $A = L + U + D$ where L, U and D are the strictly lower triangular, strictly upper triangular and diagonal parts of A, respectively. Approximate solutions to the equation can be found using iteration formula

$$\mathbf{x}^{(k+1)} = H\mathbf{x}^{(k)} + \mathbf{d},$$

where $\mathbf{x}^{(k)}$ is the kth approximation, H the iteration matrix and $\mathbf{d}$ a constant vector. $H = -D^{-1}(L + U)$ for the Jacobi method, whereas $H = -(D + L)^{-1}U$ for the Gauss-Seidel method.

(i) Consider an equation where

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} .$$

Show that the Jacobi method applied to the above equation converges, whereas the Gauss-Seidel method diverges. ***(17 marks)***

(ii) Use the Jacobi method to perform two iterations. Use $\mathbf{x}^{(0)} = (1, 2, 1)^T$ as the starting vector and work correct to four decimal places. ***(8 marks)***

End of Question Paper