The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2012–13**

Continuity and Integration **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Give the formal definition of the notion of a sequence of real numbers *converging to a limit.* ***(2 marks)***

Use the definition to show that the sequence $\left(\dfrac{n^2+1}{2n^2}\right)$ converges. ***(5 marks)***

(ii) Let (x_n) be the sequence given by $x_1 := 2014$ and

$$x_{n+1} := \frac{2220x_n}{x_n+207}, \qquad n \geq 1.$$

(a) Show that $x_n \geq 2013$ for all n.

(b) Show that (x_n) is a decreasing sequence and deduce that (x_n) converges to a limit.

(c) Find the limit of (x_n). ***(13 marks)***

2 (i) State the *Completeness Axiom.* ***(1 mark)***

Let E be a non-empty subset of $\mathbb{R}$ which is bounded below. Show that the set

$$-E := \{-x \mid x \in E\}$$

is bounded above and has a supremum. Furthermore, show that if s denotes the supremum of $-E$ then $-s$ is the infimum of E. ***(8 marks)***

(ii) State which of the statements below are true and which are false. Prove those that are true, and provide counter examples for those that are false. Theorems proved in lectures may be used without proof, provided they are precisely stated.

(a) If $a_n \to a$, $b_n \to b$ and $a_n < b_n$ for all n then $a < b$.

(b) If E is a non-empty bounded subset of $\mathbb{R}$ then $\inf E < \sup E$.

(c) It is possible for the minimum of a set to be an upper bound for that set.

(d) An increasing sequence **cannot** have a decreasing subsequence.

(e) If (x_n) and $(x_n + y_n)$ are convergent then (y_n) is convergent.

(11 marks)

3 (i) *Use sequences* to define what it means for a real-valued function f to be *continuous* at a point a in its domain. ***(2 marks)***

Prove that if the function f is continuous at the point a and the function g is continuous at $f(a)$, then the composition of f and g, $g \circ f$, defined by $g \circ f(x) := g(f(x))$ is continuous at a. ***(5 marks)***

(ii) State the *Intermediate Value Theorem* and the *Extreme Value Theorem.* ***(4 marks)***

Show that if $f : [0,1] \longrightarrow [0,2]$ is a continuous function then $f(c) = 2c$ for some $c \in [0,1]$. ***(9 marks)***

4 (i) Describe what it means for a function f to be *differentiable* at some point a in its domain. ***(2 marks)***

The function $f : [a, b] \longrightarrow \mathbb{R}$ has a minimum at $c \in (a, b)$. Show that if f is differentiable at c then $f'(c) = 0$. ***(6 marks)***

(ii) State the *Mean Value Theorem.* ***(2 marks)***

The function f is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. By considering the function $F(x) := f(x) - f(-x)$ show that

$$f(1) - f(-1) = f'(a) + f'(-a) \qquad \text{for some} \quad 0 < a < 1.$$

Also show, by considering a different auxiliary function, that

$$f(1) - 2f(0) + f(-1) = f'(b) - f'(-b) \qquad \text{for some} \quad 0 < b < 1.$$

(10 marks)

5 (i) Starting with the idea of a partition of an interval, explain what it means to say that a bounded function $f : [a, b] \longrightarrow \mathbb{R}$ is *Riemann integrable* on the interval $[a, b]$. ***(6 marks)***

State inequalities relating upper sums, lower sums, upper integral and lower integral of f. ***(2 marks)***

(ii) Let $f : [0, 2] \longrightarrow \mathbb{R}$ be given by $f(0) := 0$ and $f(x) := 1$ when $x \neq 0$, and let P_n be the partition

$$\{x_0 := 0,\ x_1 := \frac{1}{n},\ x_2 := 2\}$$

of $[0, 2]$.

(a) Sketch the graph of f and mark out a partition P_n. ***(2 marks)***

(b) Work out $U(f, P_n)$ and $L(f, P_n)$, and use your calculations to show that f is integrable on $[0, 2]$. ***(10 marks)***

End of Question Paper