Data provided: formula sheet

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2012–13**

Mathematics II (Electrical) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) You are given that

$$\mathcal{L}\{f(t)\}(s) = \frac{1}{s^3+s} \qquad \text{for } s > 0.$$

Calculate $f(t)$. **(10 marks)**

(ii) Let $f(t) \leftrightarrow F(\omega)$ be a Fourier transform pair. Compute $f * \delta(t)$ and verify that

$$\mathcal{F}\{f * \delta(t)\} = \mathcal{F}\{f\}\mathcal{F}\{\delta\}.$$

(10 marks)

2 (i) Let $f : [0, \pi] \to \mathbb{R}$ be defined by $f(t) = t$. Find the Fourier cosine series of $f(t)$. **(15 marks)**

(ii) Sketch the graph of the Fourier cosine series of $f(t)$ over the interval $[-2\pi, 2\pi]$. **(5 marks)**

3 (i) For $z = x + jy$, let

$$u(x, y) = Re(e^z) \qquad \text{and} \qquad v(x, y) = Im(e^z)$$

be the real and imaginary parts of the complex exponential function e^z. Show that

$$u_x(x, y) = v_y(x, y) \qquad \text{and} \qquad u_y(x, y) = -v_x(x, y).$$

(10 marks)

(ii) Let c be a non-zero real number, and let $f(x, y) = x^2 + cy^2$.

(a) Show that $f(x, y)$ has exactly one critical point p. **(5 marks)**

(b) For what values of c is the critical point p a local maximum/minimum/saddle? **(5 marks)**

4 (i) Let $f(x,y) = x + y - 1$ and $D \subset \mathbb{R}^2$ be the region bounded by the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$. Find

$$\iint_D f(x,y)dA.$$

(10 marks)

(ii) The set of (x,y) satisfying the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

determines an ellipse $\mathcal{E}$ in the xy-plane. Determine the area of region bounded by the ellipse $\mathcal{E}$. **(10 marks)**

Hint: perform a change of variables.

5 (i) Let $f(x,y) = x^3 + 3x^2y + 3xy^2 + y^3$.

(a) Calculate the directional derivative of $f(x,y)$ at $(\frac{1}{2}, \frac{1}{2})$ in the direction of $\mathbf{v} = (3, -4)$. **(5 marks)**

(b) In what direction is the graph of $f(x,y)$ most rapidly decreasing at the point $(\frac{1}{2}, \frac{1}{2})$ and what is the maximum rate of decrease? **(5 marks)**

(ii) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F} = ((x+y+z)^2, xyz, xy + yz + xz).$$

Calculate **div F** and **curl F**. **(10 marks)**

End of Question Paper

MAS241 FORMULA SHEET

Laplace transform:

The Laplace transform of a function $f(t)$ is given by:

$$\mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t) dt.$$

Properties of the Laplace transform: $\mathcal{L}\{f(t)\} = F(s)$ in the following table.

$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	linearity
$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$	differentiation w.r.t. t
$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$	second differentiation w.r.t. t
$\mathcal{L}\{e^{-kt}f(t)\} = F(k+s)$	frequency shift
$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s) \qquad \text{(for } a > 0)$	time shift
$\mathcal{L}\{f(at)\} = \frac{1}{a}F(\frac{s}{a}) \qquad \text{(for } a > 0)$	scaling
$\mathcal{L}\{f * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ (for $f(t), g(t)$ causal)	convolution

Table of standard Laplace transforms:

$f(t)$	$\mathcal{L}\{f(t)\}(s)$	Region of validity
$t^n \qquad \text{(for } n \geq 0)$	$\frac{n!}{s^{n+1}}$	$Re(s) > 0$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$Re(s) > 0$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$Re(s) > 0$
$H(t-T) \qquad \text{(for } T \geq 0)$	$\frac{e^{-sT}}{s}$	$Re(s) > 0$
$\delta(t-T) \qquad \text{(for } T \geq 0)$	e^{-sT}	$s \in \mathbb{C}$

Fourier transform:

The Fourier transform and inverse Fourier transforms are given by:

$$\mathcal{F}\{f(t)\}(\omega) = F(\omega) := \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt, \qquad f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega.$$

Properties of the Fourier transform: $\mathcal{F}\{f(t)\} = F(\omega)$ in the following table:

$\mathcal{F}\{e^{j\theta t}f(t)\} = F(\omega - \theta)$	frequency shift
$\mathcal{F}\{f(t-T)\} = e^{-j\omega T}F(\omega)$	time shift
$\mathcal{F}\{f^{(n)}(t)\} = (j\omega)^n F(\omega)$	differentiation
$\mathcal{F}\{F(t)\} = 2\pi f(-\omega)$	symmetry
$\mathcal{F}\{f(at)\} = \frac{1}{\lvert a\rvert}F(\frac{\omega}{a})$	scaling
$\mathcal{F}\{f * g(t)\} = \mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\}$	convolution

Table of standard Fourier transforms:

$f(t)$	$\mathcal{F}\{f(t)\}(\omega)$
$e^{-a\lvert t\rvert}$ (for $a > 0$)	$\frac{2a}{a^2+\omega^2}$
$\text{rect}_T(t)$	$\text{sinc}(\frac{T\omega}{2})$
1	$2\pi\delta(t)$

Fourier series:

The Fourier series of a periodic function $f(t)$ with fundamental period T is given by

$$S[f] = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n\cos(\omega_n t) + b_n\sin(\omega_n t)\Big)$$

where

$$\omega_n = \frac{2\pi n}{T}, \qquad a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(\omega_n t)dt, \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(\omega_n t)dt.$$

Coordinate systems:

Cylindrical polar coordinates

$$(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$$
$$(r, \theta, z) = \Big(\sqrt{x^2+y^2}, \arctan(\tfrac{y}{x}), z\Big)$$
$$dV = r dr d\theta dz.$$

Spherical polar coordinates

$$(x, y, z) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))$$
$$(\rho, \theta, \phi) = \Big(\sqrt{x^2+y^2+z^2}, \arctan(\tfrac{y}{x}), \arccos(\tfrac{z}{\rho})\Big)$$
$$dV = \rho^2\sin(\phi) d\rho d\phi d\theta.$$