The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Autumn Semester 2012 – 2013

Mathematics III (Control) **2 Hours**

Answer *four* questions. If you answer more than four questions, only your best four will be counted.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

Blank Page

1. (i) Show that the three planes $2x - y - 2z = 14$, $x - 2y - 4z = 1$, $-3x + 6y + 13z = 0$ have just one point in common, and find its coordinates.

(9 marks)

(ii) Consider the three planes $x - 2y - 4z = 1$, $-3x + 6y + 13z = 0$, $-2x + 4y + 9z = 1$.

By reducing an appropriate matrix to Hermite form, show that these planes intersect in a line and find its equation in parametric form.

(11 marks)

(iii) If the third equation in part (ii) above is replaced with $-2x + 4y + 9z = 4$, show that there are no points lying on all three planes.

(5 marks)

2. (i) Let $S_1 = \{(1,3,7), (3,-4,2), (-2,1,5)\}$ and $S_2 = \{(2,5,-3), (3,-7,4), (0,29,-17)\}$. Show that one of these sets is a basis for $\mathbb{R}^3$, but the other is not. For the set which is a basis, express the vector $(-7,23,-11)$ as a linear combination of the basis vectors. For the set which is not a basis, write one of the vectors as a linear combination of the other two vectors in the set.

(19 marks)

(ii) Find bases for the domain, kernel and image of the linear transformation

$$T(x, y, z) = (x + 5y + 3z,\ 3x + 15y - 7z)$$

so that the matrix of T with respect to these bases is in Smith form.

(6 marks)

3. Let the linear operator T be represented by the matrix

$$A = \begin{pmatrix} 4 & 7 & 6 \\ -6 & -1 & -6 \\ 0 & -6 & -2 \end{pmatrix}.$$

(i) Show that $\begin{pmatrix} 5 \\ 3 \\ -6 \end{pmatrix}$ is a fixed point of A. Express this result in terms of eigenvalues and eigenvectors.

(4 marks)

(ii) Show that 2 is an eigenvalue of A and find an associated eigenvector.

(8 marks)

(iii) Find the trace of A and hence or otherwise find a third eigenvalue of A and an associated eigenvector.

(6 marks)

(iv) Write down the characteristic polynomial of A and use the Cayley-Hamilton theorem to evaluate $A^3 - A^2 - 3A$.

(7 marks)

4. Let A be the matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & -3 \end{pmatrix}.$$

(i) Find the eigenvalues of A and state the algebraic multiplicity of each.

(4 marks)

(ii) Find the eigenspace associated with each eigenvalue and hence state the geometric multiplicity of each eigenvalue.

(8 marks)

(iii) Write down the Jordan form J of A and show that $J = P^{-1}AP$, where

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & 4 & -4 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & 1 \\ 4 & -4 & 1 \end{pmatrix}$$

(depending on how you wrote down the Jordan form).

(6 marks)

(iv) Hence, or otherwise, find the general solution of the differential equation

$$\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} = -3\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 4x.$$

(7 marks)

5. (i) Give a matrix representation of the quadratic form:

$$Q(x, y, z) = 5x^2 - 2y^2 + 11z^2 + 12xy + 12yz.$$

(4 marks)

(ii) Given that 14 is an eigenvalue, find all the eigenvalues and normalised eigenvectors for the matrix obtained in part (i).

(16 marks)

(iii) Write $Q(x, y, z)$ as a sum or difference of squares and state whether it is positive definite, negative definite or indefinite.

(3 marks)

(iv) Give a brief description of the surface $Q(x, y, z) = -28$.

(2 marks)

6. (i) Find the stationary points of the function

$$z = f(x, y) = 2x^3 + 2y^3 - 15x^2 + 15y^2 + 36x - 36y$$

and investigate their nature.

(15 marks)

(ii) Use Lagrange multipliers to find the minimum positive value of the function $f(x, y) = 5x + 8y$ on the hyperbola $x^2 - 4y^2 = 9$.

(10 marks)

End of Question Paper