INTERPLAY BETWEEN $C^*$ AND VON NEUMANN ALGEBRAS

Stuart White

University of Glasgow

26 March 2013, British Mathematical Colloquium, University of Sheffield.
### $C^*$ and von Neumann algebras

<table>
<thead>
<tr>
<th>$C^*$-algebras</th>
<th>von Neuman algebras</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Banach algebra $A$ with an involution $*$</td>
<td>- Weak operator closed $*$-subalgebra $M \subseteq B(H)$</td>
</tr>
<tr>
<td>- satisfying the $C^*$-identity $|x^*x| = |x|^2$ for all $x \in A.$</td>
<td>- Weak operator topology: $x_i \to x \Leftrightarrow \langle (x_i - x)\xi, \eta \rangle \to 0$ for all $\xi, \eta \in H$</td>
</tr>
<tr>
<td>- Norm closed $*$-subalgebra $A \subseteq B(H)$</td>
<td>- $C^*$-algebra $M$ which is isometrically the dual space of some Banach space.</td>
</tr>
<tr>
<td>Abelian $C^*$-algebras of form $C_0(X)$ for $X$ locally compact</td>
<td>Abelian vNas of form $L^\infty(X, \mu)$ for some measure space $(X, \mu)$.</td>
</tr>
</tbody>
</table>
**C* and von Neumann Algebras**

<table>
<thead>
<tr>
<th>C*-algebras</th>
<th>von Neuman algebras</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Banach algebra $A$</td>
<td>- Weak operator closed *-subalgebra $M \subseteq \mathcal{B}(\mathcal{H})$</td>
</tr>
<tr>
<td>- with an involution $*$</td>
<td>- Weak operator topology: $x_i \to x \iff \langle (x_i - x)\xi, \eta \rangle \to 0$ for all $\xi, \eta \in \mathcal{H}$</td>
</tr>
<tr>
<td>- satisfying the $C^<em>$-identity $|x^</em> x| = |x|^2$ for all $x \in A.$</td>
<td></td>
</tr>
<tr>
<td>- Norm closed *-subalgebra $A \subseteq \mathcal{B}(\mathcal{H})$</td>
<td>- $C^*$-algebra $M$ which is isometrically the dual space of some Banach space.</td>
</tr>
</tbody>
</table>

Abelian $C^*$-algebras of form $C_0(X)$ for $X$ locally compact Hff Abelian vNas of form $L^\infty(X, \mu)$ for some measure space $(X, \mu)$. 
### C* and von Neumann Algebras

#### C*-algebras
- Banach algebra \( A \)
- with an involution \(*\)
- satisfying the C*-identity
  \[ \|x^* x\| = \|x\|^2 \] for all \( x \in A \).
- Norm closed \(*\)-subalgebra \( A \subseteq \mathcal{B}(\mathcal{H}) \)

#### von Neumann Algebras
- Weak operator closed \(*\)-subalgebra \( M \subseteq \mathcal{B}(\mathcal{H}) \)
- Weak operator topology:
  \[ x_i \to x \iff \langle (x_i - x)\xi, \eta \rangle \to 0 \]
  for all \( \xi, \eta \in \mathcal{H} \)
- \( C^* \)-algebra \( M \) which is isometrically the dual space of some Banach space.

Abelian C*-algebras of form \( C_0(X) \) for \( X \) locally compact
Abelian vNas of form \( L^\infty(X, \mu) \) for some measure space \((X, \mu)\).
<table>
<thead>
<tr>
<th><strong>$C^*$-algebras</strong></th>
<th><strong>von Neumann algebras</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>- Banach algebra $A$</td>
<td>- Weak operator closed $^*$-subalgebra $M \subseteq \mathcal{B}(\mathcal{H})$</td>
</tr>
<tr>
<td>- with an involution $^*$</td>
<td>- Weak operator topology: $x_i \to x \iff \langle (x_i - x)\xi, \eta \rangle \to 0$ for all $\xi, \eta \in \mathcal{H}$</td>
</tr>
<tr>
<td>- satisfying the $C^*$-identity $|x^*x| = |x|^2$ for all $x \in A.$</td>
<td></td>
</tr>
<tr>
<td>- Norm closed $^*$-subalgebra $A \subseteq \mathcal{B}(\mathcal{H})$</td>
<td>- $C^*$-algebra $M$ which is isometrically the dual space of some Banach space.</td>
</tr>
</tbody>
</table>

Abelian $C^*$-algebras of form $C_0(X)$ for $X$ locally compact Hff

Abelian vNas of form $L^\infty(X, \mu)$ for some measure space $(X, \mu)$. 
### C* and von Neumann Algebras

<table>
<thead>
<tr>
<th>C*-algebras</th>
<th>von Neumann Algebras</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Banach algebra $A$ with an involution $*$</td>
<td></td>
</tr>
<tr>
<td>• satisfying the C*-identity $|x^* x| = |x|^2$ for all $x \in A.$</td>
<td>• Weak operator closed $^*$-subalgebra $M \subseteq B(\mathcal{H})$</td>
</tr>
<tr>
<td>• Norm closed $^*$-subalgebra $A \subseteq B(\mathcal{H})$</td>
<td>• Weak operator topology: $x_i \to x \iff \langle (x_i - x)\xi, \eta \rangle \to 0$ for all $\xi, \eta \in \mathcal{H}$</td>
</tr>
<tr>
<td>Abelian C*-algebras of form $C_0(X)$ for $X$ locally compact Hff</td>
<td>Abelian vNas of form $L^\infty(X, \mu)$ for some measure space $(X, \mu).$</td>
</tr>
</tbody>
</table>
Examples

- $M_n, \mathcal{B}(\mathcal{H})$;
- $C_0(X) \subset \mathcal{B}(L^2(X)), L^\infty(X) \subset \mathcal{B}(L^2(X))$;
- $G$ discrete group $\rightarrow$ group $C^*$ and von Neumann algebras;

$\mathcal{H} = \ell^2(G)$, basis $\{\delta_g : g \in G\}$. Define unitaries $\lambda_g : \mathcal{H} \rightarrow \mathcal{H}$ by $\lambda_g(\delta_h) = \delta_{gh}$;

$C^*_r(G) = \text{Span}\{\lambda_g : g \in G\}^\|; LG = \text{Span}\{\lambda_g : g \in G\}^\text{wot}$. 

- $\alpha : G \curvearrowright X$ Dynamical system $\rightarrow C(X) \rtimes_r G, L^\infty(X) \rtimes G$.

- $C(X) \rtimes_r G$ is a $C^*$-algebra generated by $C(X)$ and copy of $G$;
- Unitaries $(\lambda_g)_{g \in G}$ with $\lambda_{gh} = \lambda_g \lambda_h$, $\lambda_g f \lambda_g^* = f \circ \alpha^{-1}_g$ for $g, h \in G, f \in C(X)$.

How much information about $G$ or $G \curvearrowright X$ is captured by their operator algebras?
Examples

- $M_n, \mathcal{B}(\mathcal{H})$;
- $C_0(X) \subset \mathcal{B}(L^2(X)), L^\infty(X) \subset \mathcal{B}(L^2(X))$;
- $G$ discrete group $\leadsto$ group $C^*$ and von Neumann algebras;

\[ \mathcal{H} = \ell^2(G), \text{basis } \{\delta_g : g \in G\}. \text{ Define unitaries } \lambda_g : \mathcal{H} \to \mathcal{H} \text{ by } \lambda_g(\delta_h) = \delta_{gh}; \]

\[ C^*_r(G) = \operatorname{Span}\{\lambda_g : g \in G\}^\|; LG = \operatorname{Span}\{\lambda_g : g \in G\}^{\text{wot}}. \]

- $\alpha : G \curvearrowright X$ Dynamical system $\leadsto C(X) \rtimes_r G, L^\infty(X) \rtimes G$.

- $C(X) \rtimes_r G$ is a $C^*$-algebra generated by $C(X)$ and copy of $G$;
- Unitaries $(\lambda_g)_{g \in G}$ with $\lambda_{gh} = \lambda_g \lambda_h$, $\lambda_g f \lambda^*_g = f \circ \alpha^{-1}_g$ for $g, h \in G$, $f \in C(X)$.

How much information about $G$ or $G \curvearrowright X$ is captured by their operator algebras?
Another example

- Consider $M_2 \hookrightarrow M_4 \cong M_2 \otimes M_2; x \mapsto x \otimes 1$.
- This preserves the normalised trace on $M_2$ and $M_4$. Carry on and form

$$M_2 \hookrightarrow M_2 \otimes M_2 \hookrightarrow M_2^\otimes 3 \hookrightarrow \ldots$$

to obtain direct limit $^*$-algebra $\bigotimes_{n=1}^\infty M_2$.
- Perform GNS construction for canonical trace $\tau$ and take the weak operator closure — obtain hyperfinite $\text{II}_1$ factor $R$.
- Could also close in the norm topology to get the CAR-algebra $M_{2^\infty}$.
Another example

- Consider $M_2 \hookrightarrow M_4 \cong M_2 \otimes M_2; x \mapsto x \otimes 1$.
- This preserves the normalised trace on $M_2$ and $M_4$. Carry on and form
  \[
  M_2 \hookrightarrow M_2 \otimes M_2 \hookrightarrow M_2 \otimes^3 \hookrightarrow \ldots
  \]
  to obtain direct limit *-algebra $\bigotimes_{n=1}^{\infty} M_2$.
- Perform GNS construction for canonical trace $\tau$ and take the weak operator closure — obtain hyperfinite II$_1$ factor $R$.
- Could also close in the norm topology to get the CAR-algebra $M_2^{\infty}$.
Consider $M_2 \hookrightarrow M_4 \cong M_2 \otimes M_2; x \mapsto x \otimes 1$.

This preserves the normalised trace on $M_2$ and $M_4$. Carry on and form

$$M_2 \hookrightarrow M_2 \otimes M_2 \hookrightarrow M_2 \otimes^3 \hookrightarrow \cdots$$

to obtain direct limit $\ast$-algebra $\bigotimes_{n=1}^{\infty} M_2$.

Perform GNS construction for canonical trace $\tau$ and take the weak operator closure — obtain hyperfinite II$_1$ factor $R$.

Could also close in the norm topology to get the CAR-algebra $M_{2\infty}$.

**Important consequence**

$$R \cong R \overline{\otimes} R$$
**Definition**

A factor is a von Neumann algebra $M$ with trivial centre: $Z(M) = \mathbb{C}1$.

- "Every" vNa can be written as a "direct integral" of factors.
- Murray and von Neumann classified factors into types.

**Definition**

A factor $M$ is type $\text{II}_1$ if:

- it is infinite dimensional;
- $\exists$ tracial state $\tau: M \to \mathbb{C}1$, i.e. $\tau(xy) = \tau(yx), \forall x, y \in M$.

- $R$;
- $LG$ for countable discrete $G$ with $|\{hgh^{-1} : h \in G\}| = \infty$ for $g \neq e$;
- $L(X) \rtimes G$ for $G \curvearrowleft (X, \mu)$ is free, ergodic and probability measure preserving.
**Definition (Murray and von Neumann)**

A vNa $M$ is **hyperfinite** if every finite subset of $M$ can be arbitrarily approximated arbitrarily in the weak*-topology by a finite dimensional subalgebra of $M$.

- Separably acting hyperfinite vnas arise are inductive limits of finite dimensional vnas.

**Theorem (Murray and von Neumann)**

*There is a unique (separably acting) hyperfinite $\text{II}_1$ factor.*
Given two operator algebras $A, B \subseteq \mathcal{B}(\mathcal{H})$, write $d(A, B)$ for the Hausdorff distance between the unit balls of $A$ and $B$ in the operator norm.

- For a unitary $u$, $d(A, uAu^*) \leq 2\|u - 1_{\mathcal{H}}\|$. 

**Theorem (Christensen, Johnson, Raeburn-Taylor ’77)**

Let $M, N \subseteq \mathcal{B}(\mathcal{H})$ be vNas with $M$ hyperfinite.

$$d(M, N) < 1/101 \implies \exists \text{ unitary } u \in W^*(M \cup N) \text{ s.t. } uMu^* = N$$

and $\|u - 1_{\mathcal{H}}\| \leq 150d(M, N)$.

**Idea**

- Establish dimension independent perturbation results for near containments of finite dimensional algebras.

- i.e. $\forall \varepsilon > 0, \exists \delta > 0$ such that if $F$ is finite dim, $F \subseteq_{\delta} N$, then $\exists$ unitary $u \in W^*(F, N)$ with $uFu^* \subseteq N$ and $\|u - 1_{\mathcal{H}}\| < \varepsilon$. 
Given two operator algebras $A, B \subset \mathcal{B} (\mathcal{H})$, write $d(A, B)$ for the Hausdorff distance between the unit balls of $A$ and $B$ in the operator norm.

- For a unitary $u$, $d(A, uAu^*) \leq 2\|u - 1_\mathcal{H}\|$.

**Theorem (Christensen, Johnson, Raeburn-Taylor ’77)**

Let $M, N \subset \mathcal{B} (\mathcal{H})$ be vNas with $M$ hyperfinite.

\[
\begin{align*}
d(M, N) < 1/101 & \iff \exists \text{ unitary } u \in W^*(M \cup N) \text{ s.t. } uMu^* = N \\
& \text{ and } \|u - 1_\mathcal{H}\| \leq 150d(M, N).
\end{align*}
\]

**Idea**

- Establish dimension independent perturbation results for near containments of finite dimensional algebras.

- i.e. $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $F$ is finite dim, $F \subset_\delta N$, then $\exists$ unitary $u \in W^*(F, N)$ with $uFu^* \subseteq N$ and $\|u - 1_\mathcal{H}\| < \varepsilon$. 
**Definition**

C*-algebra $A$ is **AF** if every finite subset of $A$ can be arbitrarily approximated in norm by finite dimensional subalgebras of $A$.

- Separable AF algebras are inductive limits of finite dimensional algebras.
- eg $M_{2\infty}$. Changing matrix sizes matters: $M_{3\infty} \not\cong M_{2\infty}$.

**Classification — Elliott**

Separable AF-algebras are classified by $K_0$.

**Theorem (Christensen 80)**

Let $A, B \subset B(\mathcal{H})$ be C*-algebras, with $A$ separable and AF. If $d(A, B) < 10^{-9}$, then there exists a unitary $u \in W^*(A, B)$ with $uAu^* = B$.

- Don’t get $\|u - 1_\mathcal{H}\|$ small, in terms of $d(A, B)$ but can control $\text{Ad}(u) - \iota_A$ in the point-norm topology.
**Definition**

C*-algebra $A$ is AF if every finite subset of $A$ can be arbitrarily approximated in norm by finite dimensional subalgebras of $A$.

- Separable AF algebras are inductive limits of finite dimensional algebras.
- eg $M_{2^\infty}$. Changing matrix sizes matters: $M_{3^\infty} \not\cong M_{2^\infty}$.

**Classification — Elliott**

Separable AF-algebras are classified by $K_0$.

**Theorem (Christensen 80)**

Let $A, B \subseteq B(H)$ be C*-algebras, with $A$ separable and AF. If $d(A, B) < 10^{-9}$, then there exists a unitary $u \in W^*(A, B)$ with $uAu^* = B$.

- Don’t get $\|u - 1_H\|$ small, in terms of $d(A, B)$ but can control $\text{Ad}(u) - \iota_A$ in the point-norm topology.
**Definition**

C*-algebra \( A \) is **AF** if every finite subset of \( A \) can be arbitrarily approximated in norm by finite dimensional subalgebras of \( A \).

- Separable AF algebras are inductive limits of finite dimensional algebras.
- eg \( M_{2\infty} \). Changing matrix sizes matters: \( M_{3\infty} \not\cong M_{2\infty} \).

**Classification — Elliott**

Separable AF-algebras are classified by \( K_0 \).

**Theorem (Christensen 80)**

Let \( A, B \subset \mathcal{B}(\mathcal{H}) \) be C*-algebras, with \( A \) separable and AF. If \( d(A, B) < 10^{-9} \), then there exists a unitary \( u \in W^*(A, B) \) with \( uAu^* = B \).

- Don’t get \( \|u - 1_{\mathcal{H}}\| \) small, in terms of \( d(A, B) \) but can control \( \text{Ad}(u) - \iota_A \) in the point-norm topology.
Theorem (Connes 75)

Let $M$ be a (separably acting) von Neumann algebra. Then (amongst others) the following are equivalent.

1. $M$ is injective (abstract categorical property)
2. $M$ is semidiscrete (finite dimensional approximation property)
3. $M$ is hyperfinite (inductive limit structure)

Call these factors amenable.

- A linear map $\phi : A \to B$ between $C^*$-algebras is positive if $\phi(x) \geq 0$ when $x \geq 0$.
- $\phi$ is completely positive if each $\phi^{(n)} : M_n(A) \to M_n(B)$ is positive.
- $M$ is semidiscrete if $\text{id}_M$ can be approximated in the point weak* topology by completely positive contractions (cpc) $M \to F \to M$, with $F$ finite dimensional.
Amenable von Neumann algebras

Theorem (Connes 75)

Let $M$ be a (separably acting) von Neumann algebra. Then (amongst others) the following are equivalent.

1. $M$ is injective (abstract categorical property)
2. $M$ is semidiscrete (finite dimensional approximation property)
3. $M$ is hyperfinite (inductive limit structure)

Call these factors amenable.

A linear map $\phi : A \to B$ between $C^*$-algebras is positive if $\phi(x) \geq 0$ when $x \geq 0$.

$\phi$ is completely positive if each $\phi^{(n)} : M_n(A) \to M_n(B)$ is positive.

$M$ is semidiscrete if $\text{id}_M$ can be approximated in the point weak* topology by completely positive contractions (cpc) $M \to F \to M$, with $F$ finite dimensional.
Amenable von Neumann algebras

**Theorem (Connes 75)**

Let $M$ be a (separably acting) von Neumann algebra. Then (amongst others) the following are equivalent.

1. $M$ is injective (abstract categorical property)
2. $M$ is semidiscrete (finite dimensional approximation property)
3. $M$ is hyperfinite (inductive limit structure)

Call these factors amenable.

A linear map $\phi : A \rightarrow B$ between $C^*$-algebras is positive if $\phi(x) \geq 0$ when $x \geq 0$.

$\phi$ is completely positive if each $\phi^{(n)} : M_n(A) \rightarrow M_n(B)$ is positive.

$M$ is semidiscrete if $\text{id}_M$ can be approximated in the point weak* topology by completely positive contractions (cpc) $M \rightarrow F \rightarrow M$, with $F$ finite dimensional.
MORE ON CONNES THEOREM

**Corollary (Connes)**

There is a unique (separably acting) injective II$_1$ factor.

- Connes didn’t classify injective II$_1$ factors: he showed the abstract property of injectivity implies the existence of an inductive limit structure.
- Factors with this inductive limit structure had already been classified by Murray and von Neumann.

**Important Step in Connes’ Proof**

First show that an injective II$_1$ factor $M$ tensorially absorbs the hyperfinite II$_1$ factor, i.e. $M \cong M \bar{\otimes} R$.

- II$_1$ factors $M$ with $M \cong M \bar{\otimes} R$ where characterised by McDuff.
- $M \cong M \bar{\otimes} R$ iff the central sequence algebra $M^\omega \cap M'$ is non-abelian iff $M_k \hookrightarrow M^\omega \cap M'$ for all $k$. 
**Corollary (Connes)**

There is a unique (separably acting) injective II\(_1\) factor.

- Connes didn’t classify injective II\(_1\) factors: he showed the abstract property of injectivity implies the existence of an inductive limit structure.  
- Factors with this inductive limit structure had already been classified by Murray and von Neumann.

**Important Step in Connes’ Proof**

First show that an injective II\(_1\) factor \(M\) tensorially absorbs the hyperfinite II\(_1\) factor, i.e. \(M \cong M \bar{\otimes} R\).

- II\(_1\) factors \(M\) with \(M \cong M \bar{\otimes} R\) where characterised by McDuff.  
- \(M \cong M \bar{\otimes} R\) iff the central sequence algebra \(M^\omega \cap M'\) is non-abelian iff \(M_k \hookrightarrow M^\omega \cap M'\) for all \(k\).
**Definition**

A C*-algebra $A$ is **nuclear** if for every C*-algebra $B$ there is only one way of completing the algebraic tensor product $A \odot B$ to obtain a C*-algebra.

**Definition**

A C*-algebra $A$ has the **completely positive factorisation property** if $\text{id}_A : A \to A$ can be approximated by completely positive contractions $A \to F \to A$ in the point norm topology.

**Theorem (Connes, (Kirchberg, Choi-Effros))**

Let $A$ be a C*-algebra. TFAE:

1. $A$ is nuclear;
2. $A^{**}$ is semidiscrete;
3. $A$ has the completely positive approximation property.
Amenable $C^*$-algebras

**Definition**

A $C^*$-algebra $A$ is **nuclear** if for every $C^*$-algebra $B$ there is only one way of completing the algebraic tensor product $A \odot B$ to obtain a $C^*$-algebra.

**Definition**

A $C^*$-algebra $A$ has the **completely positive factorisation property** if $\text{id}_A : A \to A$ can be approximated by completely positive contractions $A \to F \to A$ in the point norm topology.

**Theorem (Connes, (Kirchberg, Choi-Effros))**

Let $A$ be a $C^*$-algebra. TFAE:

1. $A$ is nuclear;
2. $A^{**}$ is semidiscrete;
3. $A$ has the completely positive approximation property.
**Definition**

A $C^*$-algebra $A$ is **nuclear** if for every $C^*$-algebra $B$ there is only one way of completing the algebraic tensor product $A \odot B$ to obtain a $C^*$-algebra.

**Definition**

A $C^*$-algebra $A$ has the **completely positive factorisation property** if $id_A : A \to A$ can be approximated by completely positive contractions $A \to F \to A$ in the point norm topology.

**Theorem (Connes, (Kirchberg, Choi-Effros))**

Let $A$ be a $C^*$-algebra. TFAE:

1. $A$ is nuclear;
2. $A^{**}$ is semidiscrete;
3. $A$ has the completely positive approximation property.
\[ A^{**} \text{ semidiscrete} \quad \text{Hahn Banach argument} \quad A \text{ has cpap} \]

\[ A^{**} \text{ semidiscrete} \quad \text{currently requires Connes} \quad A \text{ has cpap} \]

- Due to Connes theorem, can witness semidiscreteness of \( A^{**} \) with maps \( A^{**} \rightarrow F \xrightarrow{\phi} A^{**} \) with \( \phi \) a *-homomorphism.

**Theorem (Hirshberg, Kirchberg, W.)**

Let \( A \) be a nuclear \( C^* \)-algebra. Then \( \text{id}_A : A \rightarrow A \) can be approximated by cpc maps \( A \rightarrow F \xrightarrow{\phi} A \) where \( \phi \) is a convex combination of cpc order zero maps.

- \( A \) could be projectionless: no *-hms \( F \rightarrow A \).
- Order zero maps, next best thing.
- \( \phi : F \rightarrow A \) is order zero if \( \phi \) preserves orthogonality, i.e. \( e, f \geq 0, ef = 0 \implies \phi(e)\phi(f) = 0 \).
- All cpc order zero maps \( \phi \) obtained by compressing *-homomorphism by a positive element which commutes with \( \phi(A) \).
\[ A^{**} \text{ semidiscrete} \quad \Rightarrow \quad A \text{ has cpap} \]

\[ A^{**} \text{ semidiscrete} \quad \Leftarrow \quad \text{currently requires Connes} \quad A \text{ has cpap} \]

- Due to Connes theorem, can witness semidiscreteness of \( A^{**} \) with maps \( A^{**} \to F \xrightarrow{\phi} A^{**} \) with \( \phi \) a *-homomorphism.

**Theorem (Hirshberg, Kirchberg, W.)**

Let \( A \) be a nuclear \( C^* \)-algebra. Then \( \text{id}_A : A \to A \) can be approximated by cpc maps \( A \to F \xrightarrow{\phi} A \) where \( \phi \) is a convex combination of cpc order zero maps.

- \( A \) could be projectionless: no *-homs \( F \to A \).
- Order zero maps, next best thing.
- \( \phi : F \to A \) is order zero if \( \phi \) preserves orthogonality, i.e.
  \[ e, f \geq 0, ef = 0 \implies \phi(e)\phi(f) = 0. \]
- All cpc order zero maps \( \phi \) obtained by compressing *-homomorphism by a positive element which commutes with \( \phi(A) \).
\( A^{**} \text{ semidiscrete} \quad \iff \quad A \text{ has cpap} \)

Due to Connes theorem, can witness semidiscreteness of \( A^{**} \) with maps \( A^{**} \to F \xrightarrow{\phi} A^{**} \) with \( \phi \) a \( * \)-homomorphism.

**Theorem (Hirshberg, Kirchberg, W.)**

Let \( A \) be a nuclear \( C^* \)-algebra. Then \( \text{id}_A : A \to A \) can be approximated by cpc maps \( A \to F \xrightarrow{\phi} A \) where \( \phi \) is a convex combination of cpc order zero maps.

- \( A \) could be projectionless: no \( * \)-hms \( F \to A \).
- Order zero maps, next best thing.
- \( \phi : F \to A \) is order zero if \( \phi \) preserves orthogonality, i.e.
  \( e, f \geq 0, ef = 0 \iff \phi(e)\phi(f) = 0. \)
- All cpc order zero maps \( \phi \) obtained by compressing \( * \)-homomorphism by a positive element which commutes with \( \phi(A) \).
**Theorem (Hirshberg, Kirchberg, W. 2011)**

Let $A$ be a nuclear $C^*$-algebra. Then $\text{id}_A : A \to A$ can be approximated by cpc maps $A \to F \xrightarrow{\phi} A$ where $\phi$ is a convex combination of cpc order zero maps.

**Corollary (HKW, building on results of Christensen, Sinclair, Smith, W, Winter)**

Let $A$ be separable and nuclear, and suppose that $A \subset_{\gamma} B$ for $\gamma < 1/210000$. Then $A \hookrightarrow B$.

- If in addition $d(A, B)$ small, then $\exists$ unitary $u \in W^*(A, B)$ with $uAu^* = B$.

**Definition (Winter, Zacharias, 2009)**

A $C^*$-algebra has nuclear dimension at most $n$ if one can find cp factorisations $A \xrightarrow{\text{contractive}} F \xrightarrow{\phi} A$ such that $\phi$ is a sum of at most $n + 1$ cpc order zero maps.

- $\dim_{\text{nuc}}(C(X)) = \dim(X)$; $\dim_{\text{nuc}}(A) = 0 \iff A$ is AF.
**Theorem (Hirshberg, Kirchberg, W. 2011)**

Let $A$ be a nuclear $C^*$-algebra. Then $\text{id}_A : A \to A$ can be approximated by cpc maps $A \to F \overset{\phi}{\to} A$ where $\phi$ is a convex combination of cpc order zero maps.

**Corollary (HKW, building on results of Christensen, Sinclair, Smith, W, Winter)**

Let $A$ be separable and nuclear, and suppose that $A \subset_\gamma B$ for $\gamma < 1/210000$. Then $A \hookrightarrow B$.

- If in addition $d(A, B)$ small, then $\exists$ unitary $u \in W^*(A, B)$ with $uAu^* = B$.

**Definition (Winter, Zacharias, 2009)**

A $C^*$-algebra has nuclear dimension at most $n$ if one can find cp factorisations $A \overset{\text{contractive}}{\to} F \overset{\phi}{\to} A$ such that $\phi$ is a sum of at most $n + 1$ cpc order zero maps.

- $\text{dim}_{\text{nuc}}(C(X)) = \text{dim}(X)$; $\text{dim}_{\text{nuc}}(A) = 0 \iff A$ is AF.
**Theorem (Hirshberg, Kirchberg, W. 2011)**

Let $A$ be a nuclear $C^*$-algebra. Then $\text{id}_A : A \to A$ can be approximated by cpc maps $A \to F \xrightarrow{\phi} A$ where $\phi$ is a convex combination of cpc order zero maps.

**Corollary (HKW, building on results of Christensen, Sinclair, Smith, W, Winter)**

Let $A$ be separable and nuclear, and suppose that $A \subset_\gamma B$ for $\gamma < 1/210000$. Then $A \hookrightarrow B$.

- If in addition $d(A, B)$ small, then $\exists$ unitary $u \in W^*(A, B)$ with $uAu^* = B$.

**Definition (Winter, Zacharias, 2009)**

A $C^*$-algebra has nuclear dimension at most $n$ if one can find cp factorisations $A \xrightarrow{\text{contractive}} F \xrightarrow{\phi} A$ such that $\phi$ is a sum of at most $n + 1$ cpc order zero maps.

- $\dim_{\text{nuc}}(C(X)) = \dim(X)$; $\dim_{\text{nuc}}(A) = 0 \iff A$ is AF.
Classification of separably acting injective factors.

The analogous class of $C^*$-algebras are the simple, separable
and unital $C^*$-algebras.

**Connes, Haagerup**

**Elliott’s classification programme: Initial aim**

Classify all simple, separable, nuclear $C^*$-algebras $A$ by
$K$-theory: data about homotopy equivalence classes of
projections and unitaries in matrices over $A$
Connes, Haagerup

Classification of separably acting injective factors.

The analogous class of $C^*$-algebras are the simple, separable and unital $C^*$-algebras.

Elliott’s classification programme: Initial aim

Classify all simple, separable, nuclear $C^*$-algebras $A$ by $K$-theory: data about homotopy equivalence classes of projections and unitaries in matrices over $A$
Purely infinite algebras

**Definition**

A simple $C^*$-algebra $A$ is **purely infinite** if for all $x \neq 0$ there exists $a, b \in A$ such that $1 = axb$.

- Very strong infiniteness condition. Implies that all non-zero projections are equivalent;
- Equivalently: every hereditary subalgebra has an infinite projection.
- e.g Cuntz algebras $O_n$ — universal $C^*$-algebras generated by $n$ isometries $s_1, \ldots, s_n$ with $\sum_{i=1}^n s_is_i^* = 1$.

**Theorem (Kirchberg, Kirchberg-Philips)**

Purely infinite simple separable nuclear $C^*$-algebras (satisfying the UCT) are classified by $K$-theory.
**Definition**

A simple $C^*$-algebra $A$ is purely infinite if for all $x \neq 0$ there exists $a, b \in A$ such that $1 = axb$.

- Very strong infiniteness condition. Implies that all non-zero projections are equivalent;
- Equivalently: every hereditary subalgebra has an infinite projection.
- e.g Cuntz algebras $\mathcal{O}_n$ — universal $C^*$-algebras generated by $n$ isometries $s_1, \ldots, s_n$ with $\sum_{i=1}^{n} s_is_i^* = 1$.

**Theorem (Kirchberg, Kirchberg-Philips)**

*Purely infinite simple separable nuclear $C^*$-algebras (satisfying the UCT) are classified by $K$-theory.*
### Theorem (Kirchberg)

Let $A$ be simple, separable and nuclear. Then

$$A \text{ is purely infinite } \iff A \cong A \otimes \mathcal{O}_\infty.$$ 

### Recall

To prove his theorem, Connes needed to show that an injective $\text{II}_1$ factor has $M \cong M \overline{\otimes} R$.

### Theorem (Kirchberg)

Let $A$ be simple, separable unital and nuclear. Then

$$\mathcal{O}_2 \cong A \otimes \mathcal{O}_2.$$ 

The tensorial absorption $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2$ (due to Elliott) plays a significant role in this.
Oddly, in the $C^*$-setting the purely infinite case appears to be easier than the finite case.

**Reasons**

- Higher dimensional topological phenomena can occur in simple $C^*$-algebras.
- The condition of pure infiniteness prevents this: in a precise sense these algebras have low topological dimension.

- In the finite case, need traces as well as $K$-theory.
- In late 90’s Jiang-Su constructed $\mathbb{Z}$, an infinite dimensional $C^*$-algebra with the same Elliott data as the complex numbers.
- For “reasonable” $A$, $A$ and $A \otimes \mathbb{Z}$ have same Elliott data.
- Rørdam, Toms used a Chern class obstruction of Villadsen to produce vast counter examples to Elliott’s conjecture.
Oddly, in the $C^*$-setting the purely infinite case appears to be easier than the finite case.

**Reasons**

- Higher dimensional topological phenoma can occur in simple $C^*$-algebras.
- The condition of pure infiniteness prevents this: in a precise sense these algebras have low topological dimension.

- In the finite case, need traces as well as $K$-theory.
- In late 90’s Jiang-Su constructed $\mathcal{Z}$, an infinite dimensional $C^*$-algebra with the same Elliott data as the complex numbers.
- For “reasonable” $A$, $A$ and $A \otimes \mathcal{Z}$ have same Elliott data.
- Rørdam, Toms used a Chern class obstruction of Villadsen to produce vast counter examples to Elliott’s conjecture.
Oddly, in the C*-setting the purely infinite case appears to be easier than the finite case.

**Reasons**

- Higher dimensional topological phenomena can occur in simple C*-algebras.
- The condition of pure infiniteness prevents this: in a precise sense these algebras have low topological dimension.
- In the finite case, need traces as well as $K$-theory.
- In late 90’s Jiang-Su constructed $\mathcal{Z}$, an infinite dimensional C*-algebra with the same Elliott data as the complex numbers.
- For “reasonable” $A$, $A$ and $A \otimes \mathcal{Z}$ have same Elliott data.
- Rørdam, Toms used a Chern class obstruction of Villadsen to produce vast counter examples to Elliott’s conjecture.
Establishing $\mathcal{Z}$-stability

We expect that simple, separable nuclear $C^*$-algebras should be classifiable when they absorb $\mathcal{Z}$ tensorially.

**Theorem (Winter 2010)**

Let $A$ be simple, separable unital and nuclear and have finite nuclear dimension. Then $A \cong A \otimes \mathcal{Z}$.

- Can view this as obtaining tensorial absorption from a topological property.
- Absorbing $\mathcal{Z}$ gives room to manoeuvre in classification theorems.

**Theorem (Toms, Winter 2009)**

The class $\mathcal{C} = \{ C(X) \times \mathcal{Z} : X$ finite dimensional metrisable, $\mathcal{Z} \rightarrow X$ minimal, uniquely ergodic $\}$ is classified by $K$-theory.
We expect that simple, separable nuclear $C^*$-algebras should be classifiable when they absorb $\mathcal{Z}$ tensorially.

**Theorem (Winter 2010)**

Let $A$ be simple, separable unital and nuclear and have finite nuclear dimension. Then $A \cong A \otimes \mathcal{Z}$.

- Can view this as obtaining tensorial absorption from a topological property.
- Absorbing $\mathcal{Z}$ gives room to manoeuvre in classification theorems.

**Theorem (Toms, Winter 2009)**

The class $\mathcal{C} = \{ C(X) \times \mathcal{Z} : X \text{ finite dimensional metrisable, } \mathcal{Z} \xhookrightarrow{} X \text{ minimal, uniquely ergodic} \}$ is classified by $K$-theory.
Further, $\mathcal{Z}$-stability is analogous to absorbing $R$ in a very striking way. Recall that a II$_1$ factor $M$ has $M \cong M \bar{\otimes} R$ if and only if $M_k \hookrightarrow M^\omega \cap M'$.

- For separable $A$, $A \cong A \bar{\otimes} \mathcal{Z}$ if and only if the central sequence algebra $A^\omega \cap A'$ is sufficiently non-commutative.
- For separable unital $A$, $A \cong A \bar{\otimes} \mathcal{Z}$ if and only if, for some $k \geq 2$, there exists a “large” cpc order zero map $\phi : M_k \rightarrow A^\omega \cap A'$.
- Losely, large means that $1_A - \phi(1_k)$ is dominated by $\phi(e_{11})$. 
Further, $\mathcal{Z}$-stability is analogous to absorbing $R$ in a very striking way. Recall that a $\text{II}_1$ factor $M$ has $M \cong M \bar{\otimes} R$ if and only if $M_k \hookrightarrow M^\omega \cap M'$.

- For separable $A$, $A \cong A \bar{\otimes} \mathcal{Z}$ if and only if the central sequence algebra $A_\omega \cap A'$ is sufficiently non-commutative.
- For separable unital $A$, $A \cong A \bar{\otimes} \mathcal{Z}$ if and only if, for some $k \geq 2$, there exists a “large” cpc order zero map $\phi : M_k \to A^\omega \cap A'$.
- Losely, large means that $1_A - \phi(1_k)$ is dominated by $\phi(e_{11})$. 
Further, $\mathcal{Z}$-stability is analogous to absorbing $R$ in a very striking way. Recall that a $\text{II}_1$ factor $M$ has $M \cong M \otimes R$ if and only if $M_k \hookrightarrow M^\omega \cap M'$.

- For separable $A$, $A \cong A \otimes \mathcal{Z}$ if and only if the central sequence algebra $A^\omega \cap A'$ is sufficiently non-commutative.
- For separable unital $A$, $A \cong A \otimes \mathcal{Z}$ if and only if, for some $k \geq 2$, there exists a “large” cpc order zero map $\phi : M_k \to A^\omega \cap A'$.
- Loosely, large means that $1_A - \phi(1_k)$ is dominated by $\phi(e_{11})$. 
Tensorial absorption from abstract properties

**Goal (part of a conjecture of Toms and Winter)**

Let $A$ be simple unital separable and nuclear. Then

$$A \text{ has strict comparison } \iff A \cong A \otimes \mathcal{Z}.$$ 

- $\iff$ is Rørdam.
- Strict comparison is a condition which says that we can determine the order on positive elements by looking at their ranks. The analogous condition holds for all $\text{II}_1$ factors.
- This would be a vast generalisation of Kirchberg’s $O_{\infty}$-absorption theorem as when $A$ is simple, separable and traceless Rørdam has shown:

$$A \cong A \otimes \mathcal{Z} \iff A \cong A \otimes O_{\infty}$$

$A$ purely infinite $\iff A$ has strict comparison
Tensorial absorption from abstract properties

**Goal (part of a conjecture of Toms and Winter)**

Let $A$ be simple unital separable and nuclear. Then

$$A \text{ has strict comparison } \iff A \cong A \otimes \mathcal{Z}. \quad \iff \text{ is Rørdam.}$$

- Strict comparison is a condition which says that we can determine the order on positive elements by looking at their ranks. The analogous condition holds for all $\text{II}_1$ factors.
- This would be a vast generalisation of Kirchberg’s $O_\infty$-absorption theorem as when $A$ is simple, separable and traceless Rørdam has shown:

$$A \cong A \otimes \mathcal{Z} \iff A \cong A \otimes O_\infty$$

$A$ purely infinite $\iff A$ has strict comparson
Let $A$ be simple unital separable and nuclear. Then

$$A \text{ has strict comparison } \iff A \cong A \otimes \mathcal{Z}.$$  

$\iff$ is Rørdam.

Strict comparison is a condition which says that we can determine the order on positive elements by looking at their ranks. The analogous condition holds for all $\text{II}_1$ factors.

This would be a vast generalisation of Kirchberg’s $O_\infty$-absorption theorem as when $A$ is simple, separable and traceless Rørdam has shown:

$$A \cong A \otimes \mathcal{Z} \iff A \cong A \otimes O_\infty$$

$A$ purely infinite $\iff A$ has strict comparison
Current Status

**Theorem (Kirchberg-Rørdam, Sato, Toms-W-Winter, building on Matui-Sato)**

Let $A$ be simple separable unital and nuclear and suppose that $T(A)$ is non-empty, and has a compact extreme boundary of finite covering dimension. Then

$$A \text{ has strict comparison } \iff A \cong A \otimes \mathbb{Z}.$$  

Proofs use interplay between von Neumann and $C^*$-algebras directly:

- For each extreme trace $\tau$, the weak closure $M_\tau$ of $A$ in the GNS representation is an injective $\mathrm{II}_1$ factor.
- By Connes, $M_\tau \cong M_\tau \overline{\otimes} R$ so have $M_k \hookrightarrow M_\tau^\omega \cap M'_\tau$.
- Can use the surjectivity of $A_\omega \cap A' \to M_\tau^\omega \cap M'_\tau$ to obtain order zero maps $M_k \to A_\omega \cap A'$ large wrt $\tau$.
- The condition on $T(A)$ is used to glue these together to obtain one large order zero map $M_k \to A_\omega \cap A'$. 