$K$-theory and the connection index

Roger Plymen

BMC, March 2013
David Keys [Math. Annalen] showed that the commuting algebras of certain induced representations are group algebras of the form

\[ C[R] \]
David Keys [Math. Annalen] showed that the commuting algebras of certain induced representations are group algebras of the form

\[ \mathbb{C}[R] \]

where \( R \) is one of the following groups:

\[ \mathbb{Z}/(n + 1)\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/4\mathbb{Z} \ (n \text{ odd}), \]
\[ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \ (n \text{ even}), \quad \mathbb{Z}/3\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z}, \quad 0, \quad 0, \quad 0 \]

for \( A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2 \).
David Keys [Math. Annalen] showed that the commuting algebras of certain induced representations are group algebras of the form

\[ \mathbb{C}[R] \]

where \( R \) is one of the following groups:

\[ \mathbb{Z}/(n+1)\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/4\mathbb{Z} \quad \text{(n odd)}, \]

\[ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \quad \text{(n even)}, \quad \mathbb{Z}/3\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z}, \quad 0, \quad 0, \quad 0 \]

for \( A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2 \).

Where had I seen these finite abelian groups before?
In Bourbaki, "Lie groups and Lie algebras", tables at end of chapter VI.

Let $Q(R)$ be the discrete subgroup of $V$ generated by the set $R$ of roots. Let $P(R)$ be the weight lattice in $V$ (the weights take integer values on the coroots $R^\vee$).

The above finite abelian groups are realized by $P(R)/Q(R)$ as we run through the root systems $R$ attached to $A_n, \ldots, G_2$.

**Definition**
The order of $P(R)/Q(R)$ is the connection index $\kappa$.
In Bourbaki, ”Lie groups and Lie algebras”, tables at end of chapter VI. Let $Q(R)$ be the discrete subgroup of $V$ generated by the set $R$ of roots.
In Bourbaki, "Lie groups and Lie algebras", tables at end of chapter VI. Let $Q(R)$ be the discrete subgroup of $V$ generated by the set $R$ of roots. Let $P(R)$ be the weight lattice in $V$ (the weights take integer values on the coroots $R^\vee$).

Definition The order of $P(R)/Q(R)$ is the connection index $\kappa$. 
In Bourbaki, ”Lie groups and Lie algebras”, tables at end of chapter VI. Let \(Q(R)\) be the discrete subgroup of \(V\) generated by the set \(R\) of roots. Let \(P(R)\) be the weight lattice in \(V\) (the weights take integer values on the coroots \(R^\vee\)). The above finite abelian groups are realized by \(P(R)/Q(R)\) as we run through the root systems \(R\) attached to \(A_n, \ldots, G_2\).
In Bourbaki, "Lie groups and Lie algebras", tables at end of chapter VI. Let \( Q(R) \) be the discrete subgroup of \( V \) generated by the set \( R \) of roots. Let \( P(R) \) be the weight lattice in \( V \) (the weights take integer values on the coroots \( R^\vee \)). The above finite abelian groups are realized by \( P(R)/Q(R) \) as we run through the root systems \( R \) attached to \( A_n, \ldots, G_2 \).

**Definition**

The order of \( P(R)/Q(R) \) is the connection index \( \kappa \).
Langlands duality. Given a $p$-adic group $G$ with maximal torus $T$, construct the root datum

$$(X^*(T), R, X_*(T), R^\vee)$$

Flip the roots and coroots:

$$(X_*(T), R^\vee, X^*(T), R)$$
Langlands duality. Given a $p$-adic group $G$ with maximal torus $T$, construct the root datum

$$(X^*(T), R, X^*(T), R^\vee)$$

Flip the roots and coroots:

$$(X^*(T), R^\vee, X^*(T), R)$$

**Definition**

Then $(G^\vee, T^\vee)$ is the complex connected reductive group, and maximal torus, whose root datum is the one above. $G$ will denote the maximal compact subgroup of $G^\vee$ and $T$ will denote the maximal torus of $T^\vee$. 
Langlands duality. Given a $p$-adic group $G$ with maximal torus $T$, construct the root datum

$$(X^*(T), R, X^*(T), R^\vee)$$

Flip the roots and coroots:

$$(X^*(T), R^\vee, X^*(T), R)$$

**Definition**

Then $(G^\vee, T^\vee)$ is the complex connected reductive group, and maximal torus, whose root datum is the one above. $G$ will denote the maximal compact subgroup of $G^\vee$ and $T$ will denote the maximal torus of $T^\vee$.

So, $G$ is a compact connected Lie group, $T$ a maximal torus of $G$.

**Example**

If $G = \text{SL}(n)$ then $G^\vee = \text{PGL}(n, \mathbb{C})$ and $G = \text{PU}(n) = \text{U}(n)/\text{U}(1)$. 
Thanks to Langlands duality, the set of characters of $B \subset G$ used by Keys may be identified with the compact torus $T$ in the dual group.
Thanks to Langlands duality, the set of characters of $B \subset G$ used by Keys may be identified with the compact torus $T$ in the dual group.

**Definition**

The unramified $C^*$-algebra attached to $G$ is

$$\mathcal{S}(G) = C(T, \mathbb{K})^W$$

$$= \left\{ f \in C(T, \mathbb{K}) : f(wt) = \alpha(w : t)f(t)\alpha(w : t)^{-1} \forall w \in W \right\}$$

where the Weyl group $W$ acts via the intertwining operators $\alpha(w : t)$.
Thanks to Langlands duality, the set of characters of $B \subset G$ used by Keys may be identified with the compact torus $T$ in the dual group.

**Definition**

The unramified $C^*$-algebra attached to $G$ is

$$G(G) = C(T, \mathbb{R})^W = \{ f \in C(T, \mathbb{R}) : f(wt) = a(w : t)f(t)a(w : t)^{-1} \forall w \in W \}$$

where the Weyl group $W$ acts via the intertwining operators $a(w : t)$.

What is the $K$-theory of $G(G)$?
Geometry in $\mathfrak{t}$. Let the kernel of $\exp : \mathfrak{t} \to T$ be denoted $\Gamma(T)$. We have $\Gamma(T) \cong \pi_1(T)$. The inclusion $T \to G$ induces a homomorphism $\pi_1(T) \to \pi_1(G)$ and we have a short exact sequence

$$0 \to N(G, T) \to \Gamma(T) \to \pi_1(G) \to 0$$
Geometry in $\mathfrak{t}$. Let the kernel of $\exp : \mathfrak{t} \to T$ be denoted $\Gamma(T)$. We have $\Gamma(T) \cong \pi_1(T)$. The inclusion $T \to G$ induces a homomorphism $\pi_1(T) \to \pi_1(G)$ and we have a short exact sequence

$$0 \to N(G, T) \to \Gamma(T) \to \pi_1(G) \to 0$$

We have the affine Weyl group and the extended affine Weyl group:

$$W_a = N(G, T) \rtimes W$$

$$W'_a = \Gamma(T) \rtimes W$$
Geometry in \( t \). Let the kernel of \( \exp : t \to T \) be denoted \( \Gamma(T) \). We have \( \Gamma(T) \cong \pi_1(T) \). The inclusion \( T \to G \) induces a homomorphism \( \pi_1(T) \to \pi_1(G) \) and we have a short exact sequence

\[
0 \to N(G, T) \to \Gamma(T) \to \pi_1(G) \to 0
\]

We have the affine Weyl group and the extended affine Weyl group:

\[
W_a = N(G, T) \rtimes W
\]

\[
W'_a = \Gamma(T) \rtimes W
\]

The stabilizer of an alcove \( A \) in \( t \) in \( W'_a \) is denoted \( H_A \). This can be identified with \( \pi_1(G) \). We have

\[
|H_A| = |\pi_1(G)| = \kappa.
\]
There is an affine $H_A$-equivariant retraction from $\overline{A}$ to its barycenter $x_0 \in A$.

$$r_t(x) = tx_0 + (1 - t)x$$
There is an affine $H_A$-equivariant retraction from $\bar{A}$ to its barycenter $x_0 \in A$.

$$r_t(x) = tx_0 + (1 - t)x$$

Unwrap the torus $T$ and extend the operators $a(w : t)$ to be periodic functions on $t$. We then have

$$C(T, \mathcal{R})^W = C(t, \mathcal{R})^{\Gamma(T) \times W}$$
$$= C(\bar{A}, \mathcal{R})^{H_A}$$
$$\sim_h C(x_0, \mathcal{R})^{H_A}$$
$$= \{ T \in \mathcal{R} : a(w : t_0) T = T a(w : t_0) \ \forall w \in W(t_0) \}$$

since

$$H_A = W(t_0), \quad t_0 = \exp x_0$$
Kazhdan-Lusztig parameters to the rescue!
Kazhdan-Lusztig parameters to the rescue! They are $(t_0, 1, \rho)$ where $\rho \in \text{Irr} \pi_0 C_G(t_0)$. 

Note that $C_G(t_0) = T^\vee \cdot W(t_0)$ so there are $\kappa$ such parameters. We have $S(G) \sim K \oplus K \oplus \cdots \oplus K$ with $\kappa$ copies of $K$. 

Roger Plymen

$K$-theory and the connection index
Kazhdan-Lusztig parameters to the rescue! They are

\[(t_0, 1, \rho)\]

where \(\rho \in \text{Irr} \pi_0 C_G(t_0)\). Note that

\[C_G(t_0) = T^\vee \cdot W(t_0)\]

so there are \(\kappa\) such parameters. We have

\[\mathcal{G}(\mathcal{G}) \sim \mathcal{K} \oplus \mathcal{K} \oplus \cdots \oplus \mathcal{K}\]

with \(\kappa\) copies of \(\mathcal{K}\).
Theorem

Let $\mathcal{G}$ be a split, simply connected, almost simple $p$-adic group, such as $\text{SL}(n)$. Then we have

\[
K_0(\mathcal{G}) = \mathbb{Z}^\kappa \\
K_1(\mathcal{G}) = 0
\]

where $\kappa$ is the connection index of $\mathcal{G}$. 
Theorem

Let $G$ be a split, simply connected, almost simple $p$-adic group, such as $\text{SL}(n)$. Then we have

$$K_0(S(G)) = \mathbb{Z}^\kappa$$
$$K_1(S(G)) = 0$$

where $\kappa$ is the connection index of $G$.

Intuitively, we have a deformation retraction of the $C^*$-algebra $S(G)$ onto the $L$-packet $\Pi(t_0)$ with its $\kappa$ constituents.
Baum-Connes correspondence for $\mathcal{G}$ is

$$K_{*}^{*}\left(EG\right) \simeq K_{*}\left(C^{*}_{r}(\mathcal{G})\right)$$
Baum-Connes correspondence for $G$ is

$$K_{top}^*(EG) \simeq K_*(C^*_r(G))$$

A true theorem [V. Lafforgue] —
Baum-Connes correspondence for $\mathcal{G}$ is

$$K^\text{top}_* (EG) \simeq K_* (C^*_r (\mathcal{G}))$$

A true theorem [V. Lafforgue] — but the $K$-cycles on the LHS have never been explicitly worked out (for a noncommutative group $\mathcal{G}$).
Baum-Connes correspondence for $G$ is

$$K^\text{top}_*(EG) \simeq K_*(C^r_r(G))$$

A true theorem [V. Lafforgue] — but the $K$-cycles on the LHS have never been explicitly worked out (for a noncommutative group $G$).

A refinement of Baum-Connes due to Aubert-Baum-RJP says, for example, that

$$K_*(C^*_r(G, I)) \simeq K^*_W(T)$$

the equivariant $K$-theory of the compact torus $T$ as in Atiyah’s classic book.
Baum-Connes correspondence for $\mathcal{G}$ is

$$K^\text{top}_\ast(EG) \cong K\ast(C^\ast_r(\mathcal{G}))$$

A true theorem [V. Lafforgue] — but the $K$-cycles on the LHS have never been explicitly worked out (for a noncommutative group $\mathcal{G}$).

A refinement of Baum-Connes due to Aubert-Baum-RJP says, for example, that

$$K\ast(C^\ast_r(\mathcal{G}, \mathcal{I})) \cong K^\ast_W(T)$$

the equivariant $K$-theory of the compact torus $T$ as in Atiyah’s classic book.

Essentially proved by Solleveld.
Since we have

$$\mathcal{Z}(\mathcal{G}) \hookrightarrow C^*_r(\mathcal{G}, \mathcal{I})$$

we should be able to realize the $\kappa$ generators of $K_0$ as line bundles.
Since we have
\[ \mathcal{S}(\mathcal{G}) \hookrightarrow C^*_r(\mathcal{G}, \mathcal{I}) \]
we should be able to realize the \( \kappa \) generators of \( K_0 \) as line bundles. This is done as follows: let \( \mathcal{J}_1, \ldots, \mathcal{J}_\kappa \) be an enumeration of the maximal compact subgroups of \( \mathcal{G} \), one in each conjugacy class. Let \( e(\mathcal{J}_1), \ldots, e(\mathcal{J}_\kappa) \) be their characteristic functions. These are projections in \( \mathcal{S}(\mathcal{G}) \).
Since we have

\[ \mathcal{S}(G) \hookrightarrow \mathcal{C}^*_r(G, I) \]

we should be able to realize the \( \kappa \) generators of \( K_0 \) as line bundles. This is done as follows: let \( \mathcal{J}_1, \ldots, \mathcal{J}_\kappa \) be an enumeration of the maximal compact subgroups of \( G \), one in each conjugacy class. Let \( e(\mathcal{J}_1), \ldots, e(\mathcal{J}_\kappa) \) be their characteristic functions. These are projections in \( \mathcal{S}(G) \).

**Theorem**

*The Fourier transforms* 

\[ \widehat{e}(\mathcal{J}_1), \ldots, \widehat{e}(\mathcal{J}_\kappa) \]

* determine \( W \)-equivariant line bundles on \( T \) which realize generators of \( K_0(\mathcal{S}(G)) \).*