Whitney categories and the Tangle Hypothesis

Jon Woolf, University of Liverpool
(joint with Conor Smyth)

BMC, March 2013
Small categories ‘are’ presheaves on $\Delta$ — finite ordinals and order-preserving maps — with a sheaf-like property.
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The nerve of a category

The nerve $N : \text{Cat} \rightarrow PSh(\Delta)$ is fully faithful, with essential image those simplicial sets $S$ satisfying the Segal condition

$$S_k \xrightarrow{\sim} S_1 \times_{S_0} \cdots \times_{S_0} S_1 \quad \forall \ k \in \mathbb{N}.$$  

(We think of $[k]$ as a concatenation of directed intervals

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rather than as a geometrical simplex.)
Small dagger categories ‘are’ presheaves on $D\Delta$ — finite ordinals and order-preserving or reversing maps — with a sheaf-like property.
Dagger categories (or categories with duals)

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The dagger nerve of a dagger category [Joy10]

The dagger nerve $DN: DCat \to PSh(D\Delta)$ is fully faithful, with essential image those dagger simplicial sets $S$ satisfying the Segal condition

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rather than as a sequence of arrows.)
Remark

The above realisation of $[k]$ is a typical 1d stratified space. The idea of [SW11] is to define higher categories with duals by

\[
\Delta \xrightarrow{\sim} \text{category of higher dim stratified spaces} \\
\text{Segal condition} \xrightarrow{\sim} \text{sheaf-like condition for presheaves on above}
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Remark

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\[ \text{D\Delta} \sim \text{category of higher dim stratified spaces} \]

\[ \text{Segal condition} \sim \text{sheaf-like condition for presheaves on above} \]

Definition (Higher category with duals — preliminary version)

Presheaf on a (suitable) category of stratified spaces satisfying a (suitable) sheaf-like property.
Stratified spaces

Definition (Whitney stratified manifold)
Manifold with a locally-finite partition into disjoint locally-closed submanifolds \( \{ S_i \} \) (the strata) satisfying Whitney’s condition \( B \).

Examples
A real or complex projective analytic variety admits a Whitney stratification by subvarieties. A compact manifold can be stratified by the flow of Morse–Smale vector field.

Definition (Compact cellular stratified space)
Compact union of cellular strata in a Whitney stratified manifold.

Examples
Geometric simplex \( \Delta^n \subset \mathbb{R}^{n+1} \), \( S^n \), \( \mathbb{R}P^n \), \( \mathbb{C}P^n \), Grassmannians, . . .
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Stratified and prestratified maps

Definition (Stratified map)
Smooth $f: X \to Y$, where $X$, $Y$ stratified spaces, such that
- $f^{-1} T$ is a union of strata for each stratum $T \subseteq Y$
- $f|_S: S \to T$ is a submersion for each $S \subseteq f^{-1} T$.

Definition (Prestratified map)
Smooth $f: X \to Y$ which becomes stratified after refining the stratification of $X$. 

Example
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![Diagram](https://via.placeholder.com/150)
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Whitney categories I

Definition (Categories of stratified spaces)

Objects are compact cellular stratified spaces of ambient dim $n$ and respective morphisms are

$$\text{Str}_n : \text{germs of stratified maps}$$

Definition (Whitney $n$-category)

Presheaf on $\text{hPStr}_n$ whose pullback along $\text{Str}_n / \text{Uni} \to \text{hPStr}_n$ is a sheaf. In particular

$$W(X) = \lim_{i \in S(X)} W(S_i)$$

where $S(X)$ is the poset of strata of $X$. Let $n\text{Whit}$ be the full subcategory of such presheaves.

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Example (Sheaf condition $\Rightarrow$ Segal condition)

$W(\ldots) = W(\bullet \bullet) \times_{W(\bullet)} \ldots \times_{W(\bullet)} W(\bullet \bullet)$
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Objects / morphisms

$X \leftrightarrow$ template for a pasting diagram, $W(X) \leftrightarrow$ set of pasting diagrams, or $X$-morphisms.
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Source/Target
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- Map to point
- Identities

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Structure

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- Identities
- Reflection:
- Dual

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Example (Low-dimensional cases)

$0\text{Whit} \simeq \text{Set}$ and $D\Delta \rightarrow hPStr_1$ induces $1\text{Whit} \simeq D\text{Cat}$.
Examples of Whitney categories

Example (Low-dimensional cases)

\[ \text{0Whit} \approx \text{Set} \text{ and } D\Delta \to \text{hPStr}_1 \text{ induces } \text{1Whit} \approx \text{DCat}. \]

Example (Representable Whitney categories)

Let \( \text{Rep}(X) = \text{hPStr}_n(-, X) \in n\text{Whit.} \) By Yoneda, \( \text{Rep}(X) \) is free on one \( X \)-morphism, i.e. \( n\text{Whit}(\text{Rep}(X), W) \cong W(X) \).
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Example (Framed tangles)
Define $n\text{Tang}^\text{fr}_k \in (n + k)\text{Whit}$ by

$$n\text{Tang}^\text{fr}_k (X) = \{ \text{codim } k \text{ framed sbmflds } \sqcup \text{ to strata} \} / \text{isotopy}. $$

$\in 1\text{Tang}^\text{fr}_1 ([0, 1]^2)$
Examples of Whitney categories

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Formal properties

- $n$Whit is complete and cocomplete.
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- The inclusion $n\text{Whit} \hookrightarrow \text{PSh}(h\text{PStr}_n)$ has a left adjoint.
Properties of Whitney categories

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- $n\text{Whit}$ is complete and cocomplete.
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- There is a ‘dagger nerve’ $n\text{Whit} \rightarrow \text{PSh}(D\theta_n)$ induced by $D\theta_n \rightarrow \text{hPStr}_n$ where $D\theta_1 = D\Delta$ and $D\theta_n = D\Delta \triangleright D\theta_{n-1}$. 

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### Properties of Whitney categories

#### Formal properties
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#### Definition (Category of morphisms)
For objects $w_0, w_1 \in W(pt)$ there is a Whitney $(n-1)$-category

$$W(w_0, w_1)(X) = \{\omega \in W(X \times [0,1]) : \omega|_{X \times \{i\}} = p^* w_i, i = 0, 1\}$$

of morphisms between $w_0$ and $w_1$, where $p: X \rightarrow \text{pt}$. 
The Pontrjagin–Thom construction

Choosing a generic framed point $p \in S^k$ yields a correspondence

isotopy classes of $\leftrightarrow$ homotopy classes of
framed tangles in $X$ prestratified maps $X \to S^k$
The Pontrjagin–Thom construction

Choosing a generic framed point \( p \in S^k \) yields a correspondence

\[
\text{isotopy classes of framed tangles in } X \longleftrightarrow \text{homotopy classes of prestratified maps } X \to S^k
\]

Consider \([p] \in nTang^{fr}_k(S^k)\). Since \( \text{Rep}(S^k) \) free we obtain

\[
PT: \text{Rep}(S^k) \to nTang^{fr}_k:\left[ X \xrightarrow{f} S^k \right] \leftrightarrow [f^{-1}(p) \subset X]
\]

in \((n+k)\text{Whit}\). Pontrjagin–Thom \( \Rightarrow \) \( PT \) is an isomorphism.
The Whitney Tangle Hypothesis

Definition ($k$-tuply monoidal Whitney $n$-category)

A Whitney $(n + k)$-category $W$ with $W(X) = 1$ for $\dim X < k$. 

Theorem (Whitney Tangle Hypothesis, c.f. [BD95])

$n\text{Tang} fr_k$ is the free $k$-tuply monoidal Whitney $n$-category on one $S_k$-morphism.
The Whitney Tangle Hypothesis

Definition (k-tuply monoidal Whitney n-category)
A Whitney $(n+k)$-category $W$ with $W(X) = 1$ for $\dim X < k$.

Definition (k-tuply monoidal functor)
A morphism in $(n+k)$Whit between $k$-tuply monoidal Whitney $n$-categories.
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Example

$PT: \text{Rep}(S^k) \to n\text{Tang}^\text{fr}_k$ is a $k$-tuply monoidal functor.
The Whitney Tangle Hypothesis

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References I

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