1 (i) Show that the Fourier series expansion of the function \( f(x) = \exp\left(\frac{x}{2}\right) \), defined in the interval \(-\pi \leq x \leq \pi\), is given by
\[
\frac{2 \sinh \left(\frac{\pi}{2}\right)}{\pi} + \frac{4 \sinh \left(\frac{\pi}{2}\right)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 + 1} (\cos nx - 2n \sin nx).
\]
Note: \(\sinh x = \frac{e^x - e^{-x}}{2}\).  

By setting \(x = 0\), show that
\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 + 1} = \frac{\pi - 2 \sinh \left(\frac{\pi}{2}\right)}{4 \sinh \left(\frac{\pi}{2}\right)}.
\]

(21 marks)

2 (i) Find the first four non-zero terms of the series solution of the differential equation
\[y'' - \frac{2y}{x} = 0, \quad y = y(x),\]
subject to the conditions \(y(1) = 1\) and \(y'(1) = 1\).  

(11 marks)

(ii) Use the chain rule to evaluate the value of \(dw/dt\) at \(t = 0\) given that \(w(r, s, v) = r^2 - s \tan v\) where \(r(t) = \sin^2 t\), \(s(t) = \cos t\) and \(v(t) = 4t\).  

(8 marks)

(iii) Newton’s equation, \(x^3 - 2x - 5 = 0\), has a root near \(x = 3\). Define the Newton-Raphson formula for calculating the root of a function. Starting with \(x_0 = 3\), compute \(x_1\), \(x_2\), and \(x_3\), the next three Newton-Raphson estimates for the root correct to four decimal places.  

(6 marks)
3 (i) Prove the identity
\[ \int_0^\pi \sin (nx) \sin (mx) = \begin{cases} 0, & \text{if } m \neq n \\ \pi/2, & \text{if } m = n. \end{cases} \]  
(6 marks)

(ii) Use the result of part (i) together with the method of variable separation to find the solution of the heat conduction equation
\[ \frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t), \]
subject to the boundary and initial conditions
\[ u(0,t) = 0, \quad \frac{\partial u}{\partial x}(\pi,t) = 0 \]
\[ u(x,0) = 3 \sin \left( \frac{5x}{2} \right) = f(x) \]  
(19 marks)

4 (i) Values of \( y(x) \) at \( x = 2 \) determined using the fourth-order Runge-Kutta method in conjunction with an ordinary differential equation with two different step-lengths \( h \) are given in the following table

<table>
<thead>
<tr>
<th>( h )</th>
<th>( y(2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>3.40978</td>
</tr>
<tr>
<td>0.4</td>
<td>3.39278</td>
</tr>
</tbody>
</table>

Use this data to estimate a value for \( h \) which will ensure that the error in the calculated value of \( y(2) \) using a fourth-order Runge-Kutta method does not exceed \( 10^{-4} \). Give your answer correct to 4 decimal places.  
(7 marks)

(ii) Let us consider the function
\[ f(x,y) = xe^{xy}. \]
Find all values of \( x \) such that the equality
\[ \left. \frac{\partial f(x,y)}{\partial x} \right|_{y=1} = \left. \frac{\partial f(x,y)}{\partial y} \right|_{y=1} \]
holds.  
(6 marks)

(iii) The temperature, \( T \), measured at the point \( P(x,y,z) \) of a steel beam is given in a rectangular coordinate system by
\[ T = \left[ 2x^2 + \ln(xy) + 1/z \right]^{1/2}. \]
Use the small error formula to estimate the change in temperature if the measurement is moved from the position \((6,3,2)\) to \((6.1,3.3,1.98)\). Work correct to four decimal places.  
(12 marks)

End of Question Paper
Formula sheet

• The local truncation error in the case of the 4th order Runge-Kutta method is given by
  \[ Y(x) - y(x) = Ch^4 \]
  where \( Y(x) \) is the exact value, \( y(x) \) is the estimated numerical value, \( C \) is a constant and \( h \) is the step size used in the numerical scheme.

• Chain rule
  If \( z = f(x, y) \), where \( x \) and \( y \) are both functions of \( t \), so that \( x = x(t) \) and \( y = y(t) \) we have
  \[ \frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \]
  If \( z = f(x, y) \) and both \( x \) and \( y \) are functions of \( u \) and \( v \), so that \( x = x(u, v) \) and \( y = y(u, v) \) then we have
  \[ \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \]
  \[ \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \]

• Fourier series
  If the function \( f(x) \) is defined over the interval \(-l \leq x \leq l\), then the Fourier series of \( f(x) \) is
  \[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) \]
  where
  \[ a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx, \quad (n = 0, 1, 2, \ldots) \]
  \[ b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx \quad (n = 1, 2, 3, \ldots) \]

  If the function \( f(x) \) is defined over the interval \( 0 \leq x \leq l \), then the Fourier cosine series of \( f(x) \) is
  \[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} , \quad a_n = \frac{2}{l} \int_{0}^{l} f(x) \cos \frac{n\pi x}{l} \, dx, \quad (n = 0, 1, 2, \ldots) \]
  while the sine series of \( f(x) \) is
  \[ f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} , \quad b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n\pi x}{l} \, dx \quad (n = 1, 2, 3, \ldots) \]
• Some trigonometric identities

\[
\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta
\]

\[
\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 2 \cos^2 \alpha - 1, \quad \sin 2\alpha = 2 \sin \alpha \cos \alpha
\]