1 (i) The cubic equation \( x^3 - 2x - 5 = 0 \) has a root near \( x = 2 \). Define the Newton-Raphson formula for calculating the root of a function. Starting with \( x_0 = 2 \), compute \( x_1 \), \( x_2 \), and \( x_3 \), the next three Newton-Raphson estimates for the root correct to three decimal places. (7 marks)

(ii) Let us consider the system of equations

\[
\begin{align*}
x_1 - 0.25x_2 - 0.25x_3 &= 50 \\
-0.25x_1 + x_2 - 0.25x_4 &= 50 \\
-0.25x_1 + x_3 - 0.25x_4 &= 25 \\
-0.25x_2 - 0.25x_3 + x_4 &= 25.
\end{align*}
\]

Perform three iterations using the Gauss-Seidel iterative method starting from the initial guess \([100, 100, 100, 100]^T\). Work throughout with an accuracy of three decimal places. (10 marks)

(iii) The deflection of a beam is believed to satisfy the equation

\[
\frac{d^2y}{dx^2} = e^{x^2} , \quad y = y(x),
\]

together with the boundary conditions \( y(0) = y(1) = 0 \). Estimate, using a second order finite difference approximation, the approximate deflection at 0.2, 0.4, 0.6 and 0.8. Work throughout with an accuracy of five decimal places.

**Hint:** The second order derivative of a function can be written in a finite difference form as

\[
U''_k = \frac{U_{k+1} - 2U_k + U_{k-1}}{h^2},
\]

where \( U_k \) and \( U_{k \pm 1} \) denote the values of \( U(x_k) \) and \( U(x_k \pm h) \), respectively. (8 marks)
2  

(i) Find the value of the constant \( C \), so that one of the eigenvalues of the matrix

\[
\begin{pmatrix}
2 & 4 & 3 \\
1 & 0 & 1 \\
C & 1 & 1
\end{pmatrix}
\]

is equal to -1. Further, determine the values of the remaining two eigenvalues.

(10 marks)

(ii) Fit a least squares quadratic, i.e., a polynomial of degree \( n = 2 \), to the data \((0, 5), (2, 4), (4, 1), (6, 6), (8, 7)\). The system of equations arising should be solved using Gaussian elimination with partial pivoting. Work throughout with an accuracy of four decimal places.

Hint: Assuming that the \( x_i \) values are free of errors, the normal equations used in the process of a least squares fit for a polynomial of degree \( n \) are

\[
\sum_{j=0}^{n} a_j \sum_{i=0}^{n} x_i^{j+k} = \sum_{i=0}^{n} x_i^k f_i, \quad k = 0, 1, 2, \ldots, n
\]

(15 marks)

3  

(i) Let

\[ f(x) = e^{\sin x} \]

Show that

\[
\frac{d^4 f}{dx^4} = e^{\sin x} \left( 3 + \sin x - 6 \sin x \cos^2 x - 7 \cos^2 x + \cos^4 x \right)
\]

and then evaluate

\[
\int_0^{\pi/2} e^{\sin x} dx
\]

to an accuracy of \( \epsilon = 10^{-4} \) using Simpson's method assuming that \( d^4 f/dx^4 \) takes its maximum value at one of the ends of the interval. Work throughout with an accuracy of 4 decimal places.

Hint: If a function \( f(x) \) has four continuous derivatives on an interval \((a, b)\) and this interval is divided into \( n \) subintervals, where \( n \) is an even positive integer, then the error bound for Simpson's method is given by

\[
|E_n| \leq \frac{h^4}{180} (b - a) K
\]

where

\[
h = \frac{b - a}{n}
\]

and

\[
K = \max_{a \leq x \leq b} \left| \frac{d^4 f(x)}{dx^4} \right|
\]

(13 marks)
3 (continued)

(ii) Factorise the matrix

\[
A = \begin{pmatrix}
5 & -1 & 2 \\
-10 & 4 & 3 \\
10 & 4 & -1
\end{pmatrix}
\]

into the product \( A = LU \), where \( L \) is a lower triangular matrix with unit diagonal elements and \( U \) is an upper triangular matrix.

Hence, using this factorisation, solve the matrix equation in \( x \)

\[
Ax = b
\]

where \( x = [x_1, x_2, x_3]^T \) and \( b = [2, -1, 1]^T \).

(12 marks)

4 (i) Maximize the objective function

\[
f(x, y) = 2x + 3y
\]

subject to the constraints

\[
4x + 3y \geq 12, \quad x - y \geq -3, \quad y \leq 6, \quad 2x - 3y \leq 0
\]

(12 marks)

(ii) A store wants to liquidate 200 of its shirts and 100 pairs of trousers from last season. They have decided to put together two offers, \( A \) and \( B \). Offer \( A \) is a package of one shirt and a pair of trousers which will sell for £30. Offer \( B \) is a package of three shirts and a pair of trousers, which will sell for £50. The store neither wants to sell less than 20 packages of Offer \( A \) nor less than 10 of Offer \( B \). How many packages of each do they have to sell to maximize the money generated from the promotion?

Solve the problem using a geometrical approach marking clearly the feasibility region and the line of constant revenue.

(13 marks)

End of Question Paper