Answer all four questions.
You should justify your answers carefully unless the question states otherwise.

1 (i) Write down all the units in the ring \( \mathbb{Z}_{18} \), justifying your answer. (5 marks)

(ii) What does it mean to say that a group is cyclic? Is the group \( U(\mathbb{Z}_{18}) \) cyclic? Write down its class equation. (6 marks)

(iii) Is it true that the units in the ring \( \mathbb{Z}_{18}[x] \) are the same as those in \( \mathbb{Z}_{18} \), expressed as constant polynomials? Prove this or give a counterexample. (4 marks)

Additional marks for rigour and presentation. (5 marks)

2 (i) Let \( R \) be a commutative ring. What does it mean for an element \( r \in R \) to be irreducible in \( R \)? (2 marks)

(ii) Let \( d \neq 1 \) be a square-free integer. Define \( N(r) \), the norm of an element \( r \in \mathbb{Z}[\sqrt{d}] \). Prove that if \( N(r) \) is a prime number then \( r \) is irreducible in \( \mathbb{Z}[\sqrt{d}] \). You may use the fact that \( N(st) = N(s)N(t) \). (6 marks)

(iii) Show that the following are equivalent factorisations of 5 into irreducibles in \( \mathbb{Z}[i] \):

\[
(1 + 2i)(1 - 2i) = 5 \\
(2 + i)(2 - i) = 5.
\]

Does this show that \( \mathbb{Z}[i] \) is a unique factorisation domain? (7 marks)

Additional marks for rigour and presentation. (5 marks)
3 (i) Let $f : G \rightarrow H$ be a group homomorphism. Define the kernel and image of $f$. Prove that the kernel of $\theta$ is necessarily a normal subgroup of $G$. You may assume it is a subgroup. Is the image of $f$ necessarily a normal subgroup of $H$? Prove this or give a counter-example. \hspace{1cm} (7 marks)

(ii) Recall that $D_3$ is the group of symmetries of the equilateral triangle. Consider the equilateral triangle with edges labelled as below.

```
\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0.5,0.866);
\draw (0,0) -- (-0.5,0.866);
\node at (0,0) {$\circ$};
\node at (1,0) {$\circ$};
\node at (0.5,0.866) {$\circ$};
\node at (-0.5,0.866) {$\circ$};
\end{tikzpicture}
\end{center}
```

Observe that $D_3$ acts on the numbered edges inducing a homomorphism $f : D_3 \rightarrow S_3$. Write down the kernel and image of the homomorphism $f$. Is $f$ injective? Is $f$ surjective? \hspace{1cm} (4 marks)

(iii) State, without proof, the First Isomorphism Theorem for groups. What can you deduce about the homomorphism $f$ in part (ii) by applying the First Isomorphism Theorem? \hspace{1cm} (4 marks)

Additional marks for rigour and presentation. \hspace{1cm} (5 marks)
Let $G$ be a group of order 8, with elements $a, b, c, d, e, f, g, h$, and Cayley table shown below.

$$
\begin{array}{cccccccc}
  & a & b & c & d & e & f & g & h \\
\hline
a & a & b & c & d & e & f & g & h \\
b & b & a & d & c & f & e & h & g \\
c & c & d & a & b & g & h & e & f \\
d & d & c & b & a & h & g & f & e \\
e & e & f & g & h & a & b & c & d \\
f & f & e & h & g & b & a & d & c \\
g & g & h & e & f & c & d & a & b \\
h & h & g & f & e & d & c & b & a \\
\end{array}
$$


(ii) Find a subgroup $H \subset G$, of order 2. Is $H$ a normal subgroup? (3 marks)

(iii) Write down all the left cosets $xH$. (4 marks)

(iv) Recall that the left cosets are the elements of the quotient group $G/H$. What is the order of $G/H$ in this example? Write out the Cayley table for the group $G/H$. Is $G/H$ isomorphic to the cyclic group of order 4 or the Klein 4-group? (5 marks)

Additional marks for rigour and presentation. (5 marks)

End of Question Paper