Data provided: Formula Sheet

The University Of Sheffield.

MAS323

SCHOOL OF MATHEMATICS AND STATISTICS — **Spring Semester 2012–2013**

Differential Equations: Case Studies in Applied Mathematics — **2 hours**

Attempt all questions.

1 (i) In the Canadian wilds, mountain bears eat berries and catch salmon in the rivers, and the salmon eat various insects which live near the water surface. A simple model for the bear and salmon populations, which assumes that each population exists in a finite environment, is given by *one* of the following pairs of equations for x and y:

$$\dot{x} = x(a_0 - b_0x + c_0y), \quad \dot{y} = y(a_1 - b_1y - c_1x);$$

$$\dot{x} = x(a_0 - b_0x - c_0y), \quad \dot{y} = y(a_1 + b_1y - c_1x);$$

$$\dot{x} = x(a_0 - b_0x - c_0y), \quad \dot{y} = y(a_1 - b_1y - c_1x);$$

in which the six parameters $(a_0, b_0, c_0, a_1, b_1, c_1)$ are strictly positive.

(a) Explain why the second and third pairs for $(\dot{x}, \dot{y})$ above *cannot* provide good models for the bear-fish populations, stating clearly your reasons in each case. Hence, explain why the first pair *is* a good model for the bear-fish populations and identify which populations x and y represent. ***(6 marks)***

(b) Find all the critical points of the good model and classify those three for which $xy = 0$, taking care to distinguish the separate cases $a_1b_0 > a_0c_1$ and $a_1b_0 < a_0c_1$. *Note:* You need only write down the equations satisfied for the critical point satisfying $xy \neq 0$. Do not solve them. ***(10 marks)***

(ii) Find the general solution of the autonomous system described by the equations:

$$\frac{dx}{dt} = x + 2y;$$

$$\frac{dy}{dt} = 3x + 2y.$$

State the nature of the critical point. ***(9 marks)***

2 (i) A reaction-diffusion equation describing locally the calcium-stimulated-calcium-release mechanism is

$$\frac{\partial u}{\partial t} = -A(u-u_1)(u-u_2)(u-u_3) + D\frac{\partial^2 u}{\partial x^2},$$

where $D > 0$ is the diffusion coefficient, A, u_1, u_2 and u_3 are positive constants, and $0 < u_1 < u_2 < u_3$.

(a) Using our standard technique of assuming a wave solution $u(x,t) = U(z)$ with $z = x+ct$, $c > 0$, show that $U(z)$ satisfies the second-order ordinary differential equation for the wave profile $U(z)$,

$$DU'' = cU' + A(U-u_1)(U-u_2)(U-u_3). \qquad (*)$$

Introducing $V = U'$ rewrite this equation as the system of two first-order differential equations. Find the critical points of this system.

(5 marks)

(b) You are given that any solution of the first-order equation

$$U' = -\alpha(U-u_1)(U-u_3),$$

where α is a constant, satisfies equation $(*)$. Use this condition to determine α and c. ***(10 marks)***

(ii) The Fisher equation is a model for the spread of an advantageous gene through a population. It can be written in the form

$$\frac{\partial u}{\partial t} = Au\left(1-\frac{u}{K}\right) + D\frac{\partial^2 u}{\partial x^2},$$

where u represents the population density, A and D are positive constants, and K is the carrying capacity.

(a) By rescaling the Fisher equation using

$$U = \frac{u}{K}, \qquad T = At, \qquad X = x\left(\frac{A}{D}\right)^{\frac{1}{2}},$$

show that

$$\frac{\partial U}{\partial T} = U(1-U) + \frac{\partial^2 U}{\partial X^2}. \qquad (\dagger)$$

(b) By assuming a wave solution of the form $U(X-cT)$, convert equation $(\dagger)$ into a 2nd order ordinary differential equation with independent variable $z = X - cT$.

(c) Use the substitution $V = U'$ to convert the 2nd order ordinary differential equation into a pair of 1st order ordinary differential equations. Hence find the co-ordinates of the critical points of this system.

(10 marks)

3 (i) Suppose that $x^*(t)$ is a minimising curve for the functional

$$J[x] = \int_{t_o}^{t_1} f(x, \dot{x}, t)dt$$

where $x(t_0)$ and $x(t_1)$ are fixed. Assuming, for simplicity, that the admissible curves belong to $\mathcal{C}_2$, then derive the first necessary condition

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0 \quad (*)$$

that the function $x^*(t)$ must satisfy. You may assume the result that if $g(t)$ is continuous in $[a, b]$ then $\int_a^b \eta(t)g(t)dt = 0$ for all $\eta(t)$ implies that $g(t) = 0$ for all $t \in [a, b]$. ***(8 marks)***

(ii) If t does not appear explicitly in the functional so that $f \equiv f(x, \dot{x})$, then the Euler-Lagrange equation $(*)$ given in part (a) has an explicit first integral given by

$$f - \dot{x}\frac{\partial f}{\partial \dot{x}} = \text{constant}.$$

Prove that this latter result is, in fact, a first integral of the Euler-Lagrange equations. ***(5 marks)***

(iii) Show that the function $x^*(t) = \dfrac{t}{2}$ is an extremal for the problem

$$J[x] = \int_0^2 (1 - \dot{x})^2(1 + \dot{x})^2 \; dt, \quad x(0) = 0, \quad x(2) = 1,$$

but that it is *not* a minimising curve for the problem. ***(12 marks)***

4 Consider the system of equations

$$\begin{aligned}\dot{x} &= -y - xy^2(x^2 + y^2 - 1),\\ \dot{y} &= \quad x - y^3(x^2 + y^2 - 1).\end{aligned} \qquad (*)$$

(i) Show that the coordinate origin is a critical point of this system, and it is a centre. ***(5 marks)***

(ii) Use Theorem 1 from the Formula Sheet to show that system $(*)$ does not have periodic solutions in the region $y > 1$. ***(5 marks)***

(iii) Use the variable substitution

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

to obtain the system of equations for r and θ,

$$\dot{r} = -r^3(r^2 - 1)\sin^2\theta, \qquad \dot{\theta} = 1. \qquad (\dagger)$$

Use this result to show that system $(*)$ has a limit cycle solution defined by $x^2 + y^2 = 1$. ***(6 marks)***

(iv) Linearize system $(\dagger)$ near the limit cycle solution $r = 1$. Solve this linearized system for arbitrary initial conditions. Use this solution to show that the limit cycle is asymptotically stable. (You may assume the trigonometric identity $2\sin^2\theta = 1 - \cos 2\theta$). ***(9 marks)***

End of Question Paper

List of Basic Formulae and Theorems

Theorem 1: If a periodic solution of the system of equations

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y)$$

exists in a simply connected region, then $f_x + g_y = 0$ somewhere in that region.

Corollary: There are no periodic solutions in any simply connected region where $f_x + g_y \neq 0$ everywhere.

Theorem 2: The orbit $\mathcal{C}$ of a periodic solution must enclose at least one critical point.

Orthogonality conditions for trig functions

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad \text{when } m \neq n.$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0.$$

Extremals of functional

$$J[y] = \int_{x_0}^{x_1} f(y, y', x) \, dx$$

are the solutions to the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$