1 For the equation \( y'(x) = f(x, y(x)) \), the AB3 method is defined as

\[
y_{n+1} = y_n + \frac{1}{12} h(23f_n - 16f_{n-1} + 5f_{n-2}),
\]

and the AM2 method is defined as

\[
y_{n+1} = y_n + \frac{1}{12} h(5f_{n+1} + 8f_n - f_{n-1}),
\]

where \( f_n = f(x_n, y_n) \) according to the usual notations. For the AB3 method, the local truncation error is \( T_P = \frac{3}{8} h^4 y^{(4)}(\xi_1) \), while for AM2 it is \( T_C = -\frac{1}{24} h^4 y^{(4)}(\xi_2) \), where \( x_{n-2} \leq \xi_1 \leq x_{n+1} \) and \( x_{n-1} \leq \xi_2 \leq x_{n+1} \).

(i) Given the differential equation and initial condition

\[
y'(x) = -3y^2 \sin(x), \quad y(0) = 0.5,
\]

and the values \( y(0.1) = 0.4963 \) and \( y(0.2) = 0.4855 \), apply the ABM method (with AB3 as the predictor and AM2 as the corrector) to find the approximate solution at \( x = 0.3 \), using step size \( h = 0.1 \). Work throughout correct to four decimal places (note the argument of \( \sin(x) \) should be understood in radians). (8 marks)

(ii) Estimate the local truncation error for the approximate solution at \( x = 0.3 \) using Milne’s device. (10 marks)

(iii) Show that the AM2 method is convergent. (7 marks)
A single step method for equation \( y' = f(x, y) \) is defined by the following formulas:

\[
k_1 = hf_n, \quad k_2 = hf\left(x_n + \frac{5}{6}h, y_n + \frac{5}{6}k_1\right),
\]

\[
y_{n+1} = y_n + \frac{2}{5}k_1 + \frac{3}{5}k_2,
\]

where \( f_n = f(x_n, y_n) \).

(i) Find the interval of absolute stability for the method when it is applied to the test equation \( y' = \lambda y \), where \( \lambda \) is a constant. \( (10 \text{ marks}) \)

(ii) Show that the method is at least of order 2. You may use the formula for the Taylor expansion of a function \( f(x, y) \):

\[
f(x + h, y + k) = f(x, y) + h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y} + \frac{1}{2} \left( h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2} \right) + \ldots
\]

where \( h \) and \( k \) are constants. \( (15 \text{ marks}) \)

Laplace’s equation for the function \( u(r, \theta) \) in polar coordinates \((r, \theta)\) is

\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.
\]

(i) Considering a separable solution \( u(r, \theta) = R(r)\Theta(\theta) \), show that

\[
r^2 \frac{R''}{R} + r \frac{R'}{R} = -\frac{\Theta''}{\Theta} = \alpha.
\]

Explain why \( \alpha \) must be a constant. \( (6 \text{ marks}) \)

(ii) Given that there are only trivial solutions when \( \alpha \leq 0 \), apply the periodic boundary conditions to find \( \alpha \), hence find the solution to \( u(r, \theta) \) for \( r > a \), subject to the conditions

\[
\left. \frac{\partial u(r, \theta)}{\partial r} \right|_{r=a} = u_0 \cos \theta, \quad \text{and} \quad u(r, \theta) \to 0, \quad \text{when} \quad r \to \infty.
\]

(19 marks)

Find the solution \( u(x, y) \) to the following second order hyperbolic partial differential equation

\[
2 \frac{\partial^2 u}{\partial x^2} + 3 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 2,
\]

subject to conditions

\[
u(x, 0) = e^{-x^2}, \quad \left. \frac{\partial u(x, y)}{\partial y} \right|_{y=0} = 0, \quad (-\infty < x < +\infty).
\]

Hint: start with suitable variable substitutions. \( (25 \text{ marks}) \)